ESE 2023

Main Exam Detailed Solutions

Electrical Engineering

PAPER-I

EXAM DATE : 25-06-2022 | 09:00 AM to 12:00 PM

MADE EASY has taken due care in making solutions. If you find any discrepency/error/typo or want to contest the solution given by us, kindly send your suggested answer(s) with detailed explanation(s) at: **info@madeeasy.in**

ANALYSIS

Electrical Engineering Paper-I
ESE 2023 Main Examination

Sl.	Subjects	Marks
1.	Electric Circuits	64
2.	Electromagnetic Fields	52
3.	Electrical Materials	64
4.	Engineering Mathematics	104
5.	Basic Electronics Engineering	72
6.	Computer Fundamental	32
7.	Electrical and Electronic Measurements	92
		Total 480

Scroll down for detailed solutions

SECTION : A

Q.1 (a) Suppose A is a 3×3 diagonalizable matrix. Then
(i) show that that each eigen value of A is 0 or 1, if $A^2 = A$;
(ii) find the trace of the matrix $B = A + A^3 + A^{-1}$, if the eigen values of A are 2, 3, –2.

[12 marks : 2023]

Solution:

(i) As the matrix A is diagonalizable, there exists a non-singular matrix P, such that $P^{-1}.A.P = D$, where D is a diagonal matrix whose diagonal elements are the eigen values of A.

Thus, $D = \text{diagonal}\{d_1 \,.....\, d_n\}$

As eigen values are 0 or 1 only.

We see that $(d_i)^2 = (d_i)$; for every i

That implies that

$$D^2 = \text{diagonal}\{(d_1)^2, \,......\, (d_n)^2\}$$
$$= \text{diagonal}\{(d_1), \,.....\, (d_n)\}$$
$$= D$$

Hence, $A^2 = [PDP^{-1}]^2 = PDP^{-1} \,.\, PDP^{-1}$
$$= PDIDP^{-1} = PD^2P^{-1} = PDP^{-1} = A$$

Therefore, $A^2 = A$

(ii) Given eigen values of A are 2, 3, –2.

So, eigen values of B corresponding to 2.

$$B = A + A^3 + A^{-1}$$
$$= 2 + 2^3 + \frac{1}{2}$$
$$= 2 + 8 + \frac{1}{2} = \frac{21}{2}$$

Eigen values of B corresponding 3.

$$B = A + A^3 + A^{-1}$$
$$= 3 + 3^3 + \frac{1}{3}$$
$$= 3 + 27 + \frac{1}{3} = \frac{9 + 81 + 1}{3}$$
$$= \frac{91}{3}$$

Eigen values of B corresponding-2

$$B = A + A^3 + A^{-1}$$
$$= -2 + (-2)^3 + \frac{1}{-2}$$

$$= -2 - 8 - \frac{1}{2}$$

$$= -\frac{21}{2}$$

End of Solution

Q.1 (b) Evaluate the linear density in atoms per mm in the following directions in BCC iron, which has lattice constant of 2.89 Å :

(i) [1 0 0]

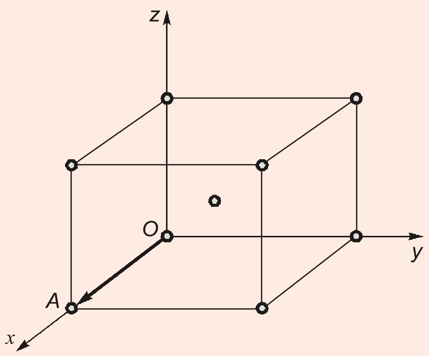
(ii) [1 1 0]

(iii) [1 1 1]

[12 marks : 2023]

Solution:

(i) BCC Unit Cell :



Vector *OA* is along [1 0 0]

Length of vector *OA* = Edge length of cube

$= a = 2.89$ Å

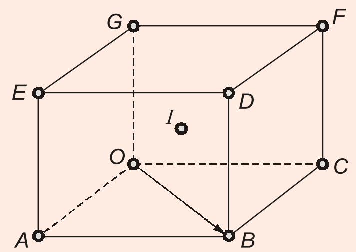
No. of atoms centred on *OA* $= \frac{1}{2} + \frac{1}{2} = 1$ atm

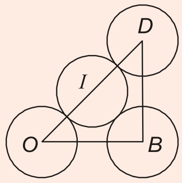
So, linear density along *OA*

i.e., $$LD_{[100]} = \frac{1 \text{ atom}}{\text{Length of } OA} = \frac{1 \text{ atom}}{2.89 \times 10^{-7} \text{ mm}}$$

$\Rightarrow$ $LD_{[100]} = 3.46 \times 10^6$ atoms/mm

(ii) Vector *OB* is along [1 1 0]





$$OD = 4R$$
$$BD = a$$

For BCC, $a = \frac{4R}{\sqrt{3}}$

$\therefore$ $OB = \sqrt{OD^2 - BD^2} = \sqrt{16R^2 - \frac{16R^2}{3}} = 4R\sqrt{\frac{2}{3}}$

$$LD_{[110]} = \frac{\text{No. of atoms centred on [110] direction}}{\text{Length of [110] direction vector}}$$

$$= \frac{\left(\frac{1}{2}+\frac{1}{2}\right)\text{atom}}{OB} = \frac{1 \text{ atom}}{4R \times \sqrt{\frac{2}{3}}}$$

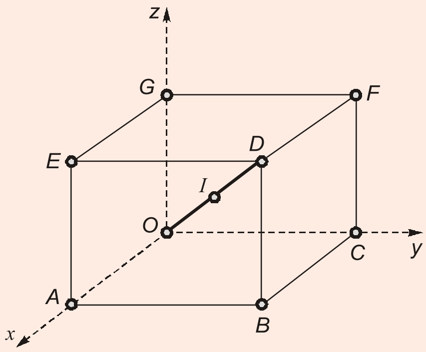
But $R = \frac{\sqrt{3}a}{4}$

$\therefore$ $LD_{[110]} = \frac{1 \text{ atom}}{4 \times \frac{\sqrt{3}a}{4} \times \frac{\sqrt{2}}{\sqrt{3}}} = \frac{1 \text{ atom}}{\sqrt{2}a}$

$\Rightarrow$ $LD_{[110]} = \frac{1 \text{ atom}}{\sqrt{2} \times 2.89 \times 10^{-7} \text{mm}} = 2.446 \times 10^6$ atoms/mm

$\Rightarrow$ $LD_{[110]} = 2.446 \times 10^6$ atoms/mm

(iii) OD is [111] direction.



$$OD = 4R$$

$\Rightarrow$ $OD = 4 \times \frac{\sqrt{3}a}{4}$

$$OD = \sqrt{3}a$$

$$\text{Number of atoms entered on [1 1 1]} = \frac{1}{2} + 1 + \frac{1}{2} = 2$$

$$\therefore \quad LD_{[111]} = \frac{2 \text{ atoms}}{\sqrt{3}a} = \frac{2 \text{ atoms}}{\sqrt{3} \times 2.89 \times 10^{-7} \text{mm}}$$

$$= 3.99 \times 106 \text{ atoms/mm}$$

$$\Rightarrow \quad LD_{[111]} \simeq 4 \times 10^6 \text{ atoms/mm}$$

End of Solution

Q.1 (c) Derive an expression for capacitance (C) of concentric spheres having radii a and b ($a < b$) respectively with single dielectric.

[12 marks : 2023]

Solution:

$$V = \int \bar{E} \cdot \overline{dl}$$

$$\bar{E}\Big|_{\text{sphere}} = \frac{Q}{4\pi \in r^2} \hat{a}_r$$

$$\overline{dl} = dl \hat{a}_r$$

$$\therefore \quad V = -\int_a^b \frac{Q}{4\pi \in r^2} dr = \frac{Q}{4\pi \in} \frac{1}{r}\bigg|_a^b = \frac{Q}{4\pi \in}\left[\frac{1}{b} - \frac{1}{a}\right]$$

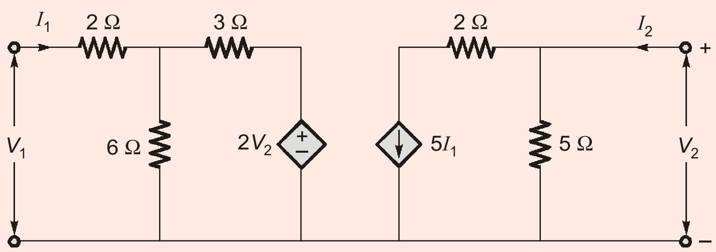
$$= -\frac{Q}{4\pi \in}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Negative sign signifies that in the direction of $\overline{dl}$ from $a \to b$, V decreases. Hence, taking absolute value of V, we get

$$C = \frac{Q}{|V|} = \frac{4\pi \in}{\frac{1}{a} - \frac{1}{b}} \text{ F}$$

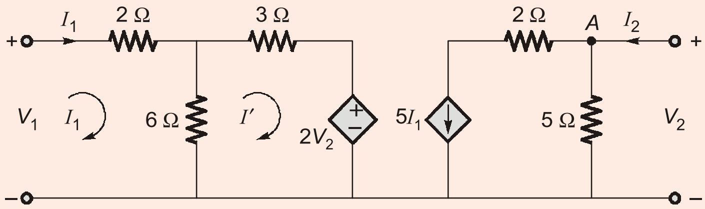
End of Solution

Q.1 (d) Find the hybrid parameters of the following circuit :



[12 marks : 2023]

Solution:



Find hybrid parameter of the network shown,

$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$I_1 = h_{21}I_1 + h_{22}V_2$$

KCL at A,

$$5I_1 + \frac{V_2}{5} = I_2$$
$$h_{21}I_1 + h_{22}V_2 = I_2$$
$$\left.\right\} \quad h_{21} = 5, \quad h_{22} = 1/5\,℧$$

By KVL,

$$V_1 = (2+6)I_1 - 6I' \qquad \text{...(i)}$$
$$(3+6)I' - 6I_1 + 2V_2 = 0$$
$$I' = \frac{6I_1 - 2V_2}{9} \qquad \text{...(ii)}$$

Substitute equation (ii) in (i),

$$V_1 = 8I_1 - 6\left(\frac{6I_1 - 2V_2}{9}\right)$$
$$= 8I_1 - \frac{12}{3}I_1 + \frac{4}{3}[V_2]$$

$\Rightarrow$
$$V_1 = 4I_1 + \frac{4}{3}V_2$$

We know,
$$V_1 = h_{11}I_1 + h_{12}V_2$$
$$\left.\right\} \quad h_{11} = 4\ \Omega, \quad h_{12} = 4/3$$

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 4 & \frac{4}{3} \\ 5 & \frac{1}{5} \end{bmatrix}$$

End of Solution

Q.1 (e) Construct full :

(i) conjunctive normal form for the statement $P \rightarrow Q$;

(ii) disjunctive normal form for the statement $(P \rightarrow \{Q \vee R\}) \wedge (P \vee Q)$.

[6 + 6 = 12 marks : 2023]

Solution:

To construct the full Conjunctive Normal Form (CNF) for the implication P → Q, we need to convert it into a series of conjunctions. Here's the process:

Step 1: Negate P and rewrite the implication as ¬P ? Q.
Step 2: Convert ¬P ? Q into CNF form.
To convert ¬P ? Q into CNF form, we can apply the distributive law:
(¬P ? Q) is equivalent to (¬P ? Q) ? (P ? Q).
Therefore, the CNF form for P -> Q is:
(¬P ? Q) ? (P ? Q).
To construct the full disjunctive normal form (DNF) for the given formulas, let's first break down each formula into its constituent parts:
Formula 1: (P -> (Q v R))
Formula 2: (P v Q)
Now, let's convert each formula into its DNF form separately:
DNF for Formula 1:
Step 1: Apply the implication equivalence (P -> Q) = (~P v Q) to the main implication:
(~P v (Q v R))
DNF for Formula 2:
This formula is already in disjunctive form.
Now, we need to combine the two formulas to obtain the full DNF.
To combine them, we'll distribute Formula 2 over Formula 1:
(P v Q) ? (~P v (Q v R))
Now, we can expand the conjunction to get all the possible disjunctive clauses:
(P ? ~P) v (P ? (Q v R)) v (Q ? ~P) v (Q ? (Q v R))
Simplifying further:
False v (P ? (Q v R)) v (Q ? ~P) v (Q ? (Q v R))
Since False v anything = anything, we can simplify the expression:
(P ? (Q v R)) v (Q ? ~P) v (Q ? (Q v R))
This is the full disjunctive normal form for the given formulas.

End of Solution

Q.2 (a) (i) Obtain the half-range cosine series for the funtion $f(x) = \sin x$ in $0 \le x \le \pi$ and hence, find the value of $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

(ii) Evaluate the integral $\iint_R (x-y)^4 \cos^2(x+y)\,dx\,dy$ where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$, $(0, \pi)$.

[10 + 10 = 20 marks : 2023]

Solution:

(i) Half range cosine series is given by

$$a_o = \frac{1}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\,; \quad b_n = 0$$

Here, $f(x) = \sin x$

Range : $(0, \pi)$, i.e., $l = \pi$

$\therefore$ $$f(x) = a_o + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_o = \frac{1}{\pi}\int_0^\pi f(x)dx$$

$\therefore$ $$a_o = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi}$$

$\therefore$ $$a_o = \frac{2}{\pi}$$

Now, $$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi [\sin(1+n)x + \sin(1-n)x]dx$$

$$a_n = \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{(1+n)} - \frac{\cos(1-n)x}{(1-n)}\right]_0^\pi$$

$$a_n = \frac{1}{\pi}\left[\frac{-\cos(\pi+\pi n)}{1+n} - \frac{\cos(\pi-\pi n)}{1-n} + \frac{1}{(1+n)} + \frac{1}{(1-n)}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos\pi n}{(1+n)} - \frac{\cos\pi n}{(1-n)} + \frac{2}{1-n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\cos\pi n \times \left(\frac{2}{1-n^2}\right) + \frac{2}{1-n^2}\right]$$

$\therefore$ $$a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{\pi}\cdot\frac{4}{(1-n^2)}, & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots..\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{4x^2-1} = \frac{1}{2}\sum_{n=1}^{K}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

For $K = 1, 2, \ldots..$

$$S_1 = \frac{1}{2}\sum_{n=1}^{1}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right)$$

$$S_2 = \frac{1}{2}\sum_{n=1}^{2}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)\right] = \frac{1}{2}\left(1-\frac{1}{5}\right)$$

$$S_3 = \frac{1}{2}\sum_{n=1}^{3}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\right] = \frac{1}{2}\left(1-\frac{1}{7}\right)$$

$$S_K = \frac{1}{2}\left(1-\frac{1}{2K+1}\right)$$

$$\lim_{K\to\infty} S_K = \frac{1}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \lim_{K\to\infty} S_K = \frac{1}{2}$$

(ii) Translate rhombus to origin :

$$x' = x-\pi,\ y' = y-\pi$$

$$= \iint_R (x-y)^4\cos^2(x+y)dxdy$$

$$= \iint_{R'} (x'-y')^4\cos^2(x'+y'+2\pi)dx'dy'$$

Change coordinate system :

$$x' - y' = u,\ x' + y' = v$$

Now, $-\pi \le u \le \pi,\ -\pi \le v \le \pi$

$$\iint_{R'} (x'-y')^4\cos^2(x'+y')dx'dy' = \iint_{R''} u^4\cos^2(v)J(u,v)dudv$$

Compute Jacobian : $J = \frac{1}{2}$

$$\frac{1}{2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} u^4\cos^2(v)dudv = \frac{1}{2}\int_{-\pi}^{\pi}u^4du\int_{-\pi}^{\pi}\cos^2(v)dv$$

$$= \frac{1}{2}\cdot\frac{u^5}{5}\Big|_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos^2(v)dv$$

$$= \frac{1}{2}\left(\frac{\pi^5}{5}+\frac{\pi^5}{5}\right)\pi$$

$$= \frac{1}{2}\times\frac{2\pi^5}{5}\times\pi$$

$$= \frac{\pi^6}{5}$$

End of Solution

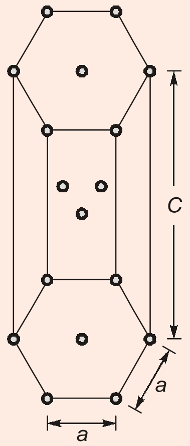
Q.2 (b) (i) Determine the volume of an HCP unit cell in terms of its *a* and *c* lattice parameters.

(ii) Copper has an atomic radius of 0.13 nm, an FCC crystal structure and an atomic weight of 63.5 g/mol. Evaluate its theoretical density and compare the answer with its measured density. (Take Avogadro number, N_A = 6.022 × 10^{23} atoms/mol).

[10 + 10 = 20 marks : 2023]

Solution:

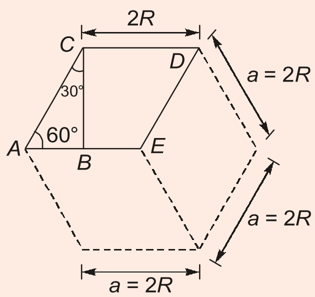
(i) HCP unit cell is shown below :



$$\frac{c}{a} = 1.633 \text{ for HCP}$$

Volume (V_c) of HCP unit cell = Base area × Cell height (c)

⇒ To calculate base area consider following figure :



$$BC = 2R\cos 30° = \frac{2R\sqrt{3}}{2}$$

⇒ $$BC = \sqrt{3}R$$

$$R = 2R \quad \Rightarrow \quad R = \frac{9}{2}$$

The area of $ACDE$ = $CD \times BC = 2R \times \sqrt{3}R$

The base area is just three times the area of the parallelpied *ACDE* shown above.

∴ Area of base = 3 × Area of $ACDE$ = $3 \times 2R \times \sqrt{3}R$

$$= 6\sqrt{3}R^2 = 6\sqrt{3}\times\left(\frac{9}{2}\right)^2 = \frac{6\sqrt{3}}{4}a^2$$

Hence, volume of unit cell

$$V_c = \text{Area}\times C = \frac{6\sqrt{3}}{4}a^2\times C$$

$$V_c = 2.598a^2C$$

(ii) Theoretical density,

$$\rho = \frac{nA_{Cu}}{V_cN_A}$$

where, n = Number of atoms per unit cell

A_{Cu} = Atomic weight of Cu

V_c = Volume of unit cell

$$V_c = a^3 = \left(2\sqrt{2}R\right)^3$$

$\Rightarrow$ $$V_c = 16\sqrt{2}R^3$$

$\therefore$ $$\rho = \frac{4\times 63.5}{16\sqrt{2}(0.13\times10^{-7})^3\times6.023\times10^{23}}\ \text{g/cm}^3$$

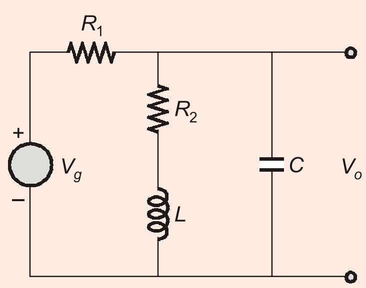
ρ = 8.48 g/cm^3 (Theoretical density of Cu)

The literature value (measured value) for the density of Cu is 8.94 g/cm^3, which is in very close agreement with above calculated value.

End of Solution

Q.2 (c) (i) Two inductive coils having same self-inductance when connected in series carrying a current of I amperes store W joules of magnetic energy in their fields. When the connections of one of the coils are interchanged and the current is reduced to $\left(\frac{I}{3}\right)$ amperes, the stored energy remains the same. Calculate the ratio of mutual to self-inductance.

(ii) Determine the transfer function $\frac{V_o(s)}{V_g(s)}$ for the circuit shown below for R_1 = 500 Ω, R_2 = 50 Ω, L = 10 mH and C = 2 μF :

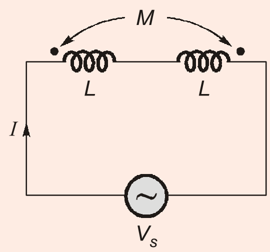


[10 + 10 = 20 marks : 2023]

Solution:

(i) Two inductive coils having same self inductance when connected in series carrying a current of I amp and stores energy W Joule. When the connection of one of the coils interchanged, then current is reduced to $\frac{I}{3}A$. The energy stored is remains same calculate ratio of mutual and self inductance.

Series opposing

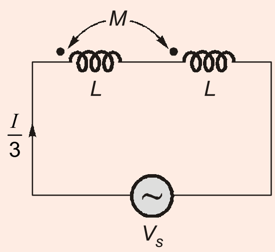


$$L_{eq} = L_1 + L_2 - 2M$$

$$W = \frac{1}{2}L_{eq} \cdot I^2$$

$$W = \frac{1}{2}[L + L - 2M]I^2 \quad \text{...(i)}$$

Series aiding



$$L_{eq} = L_1 + L_2 + 2M$$

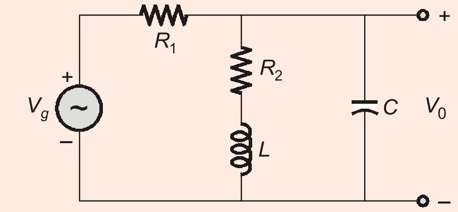
$$W = \frac{1}{2}L_{eq}(I')^2$$

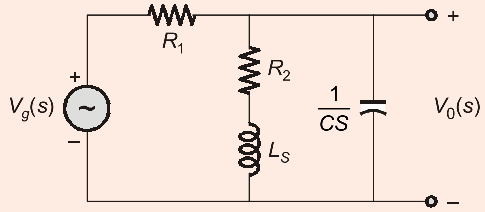
$$W = \frac{1}{2}[L_1 + L_2 + 2M]\left(\frac{I}{3}\right)^2 \quad \text{...(ii)}$$

By equating (i) = (ii),

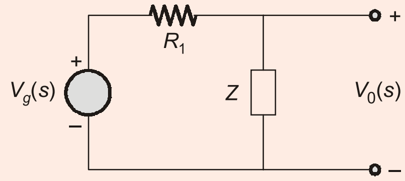
$$\frac{M}{L} = \frac{4}{5}$$

(ii)





$$Z'(s) = \frac{(R_2 + Ls)\cdot\frac{1}{Cs}}{R_2 + Ls + \frac{1}{Cs}} \Rightarrow \frac{R_2 + Ls}{LCs^2 + R_2Cs + 1}$$



$$V_g(s) = [R, Z'(s)]I(s) \quad \text{...(1)}$$

$$V_o(s) = Z'(s)I(s) \quad \text{...(2)}$$

Dividing equation (2) by equation (1)

$$\frac{V_o(s)}{V_g(s)} = \frac{\frac{R_2 + Ls}{LCs^2 + R_2Cs + 1}}{R_1 + \frac{R_2 + Ls}{LCs^2 + R_2Cs + 1}}$$

$$\frac{V_o(s)}{V_g(s)} = \frac{R_2 + Ls}{R_1LCs^2 + R_1R_2Cs + R_1 + R_2 + Ls}$$

$$\frac{V_o(s)}{V_g(s)} = \frac{R_2 + Ls}{R_1LCs^2 + (R_1R_2C + L)s + R_1 + R_2}$$

$$\frac{V_o(s)}{V_g(s)} = \frac{50 + 10^{-2}s}{10^{-5}s^2 + 0.06s + 550}$$

$$= \frac{10^{-2}[s + 5000]}{10^{-5}[s^2 + 6000s + 55\times10^6]}$$

$$\frac{V_o(s)}{V_g(s)} = \frac{1000(s + 5000)}{s^2 + 6000s + 55\times10^6}$$

End of Solution

Q.3 (a) (i) Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x \partial y}-6\frac{\partial^2 z}{\partial y^2} = e^{x+2y} + \sin(4x + 3y) + y\cos x$$

(ii) Compute the following integral by residue theorem :

$$\int_0^{2\pi} \frac{\sin\theta}{3-2\sin\theta} d\theta$$

[10 + 10 = 20 marks : 2023]

Solution:

(i) Its auxillary equation is $m^2 + m - 6 = 0$

$\Rightarrow$ $m = -3, 2$

$\therefore$ Complementary function :

$$CF = f_1(y-3x) + f_2(y+2x)$$

Find PI :

$$\text{P.I.} = \frac{e^{2y+x}}{D^2+DD'-6D'^2}+\frac{\sin(3y+4x)}{D^2+DD'-6D'^2}+\frac{y\cos x}{D^2+DD'-6D'^2}$$

$$= \text{P.I.}_1 + \text{P.I.}_2 + \text{P.I.}_3$$

P.I.$_1$:

$$\text{P.I.} = \frac{e^{2y+x}}{D^2+DD'-6D'^2} = \frac{e^{2y+x}}{1^2+1\times 2-6\times 2^2}$$

$$= \frac{e^{2y+x}}{-21}$$

P.I.$_2$:

$$\text{PI}_2 = \frac{\sin(3y+4x)}{D^2+DD'-6D'^2}$$

Put $D^2 = -4^2$

$$DD' = -4\times 3,\ D'^2 = -3^2$$

$$\text{PI}_2 = \frac{\sin(3y+4x)}{-4^2-4\times 3-6(-3^2)} = \frac{\sin(3y+4x)}{26}$$

$$\text{PI}_3 = \frac{y\cos x}{D^2+DD'-6D'^2}$$

$$= \frac{1}{(D-2D')(D+3D')}y\cos x$$

$$= \frac{1}{D-2D'}\int (C+3x)\cos x\,dx$$

$(\because\ y = C - mx = C + 3x)$

$$= \frac{1}{D-2D'}[(C+3x)\sin x+3\cos x]_{C\to y-3x}$$

$$= \frac{1}{D-2D'}(y\sin x+3\cos x)$$

$$= \int ((C-2x)\sin x+3\cos x)dx$$

$$= (C - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x$$

Put $y = C - 2x$

$$PI_3 = -y \cos x + \sin x$$

$\therefore$ General solution of P.D.E. is

$$Z = CF + PI$$

$$Z = f_1(y - 3x) + f_2(y + 2x) + \frac{e^{2y+x}}{-21} + \frac{\sin(3y + 4x)}{26} + \sin x - y\cos x$$

(ii) Given : $\int_0^{2\pi} \frac{\sin\theta}{3 - 2\sin\theta} \cdot d\theta$

$\because$ $$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$= \int_0^{2\pi} \frac{e^{i\theta} - e^{i\theta}}{6i - 2(e^{i\theta} - e^{i\theta})} d\theta$$

$\because$ $$e^{i\theta} = z$$

$$i e^{i\theta}.d\theta = dz$$

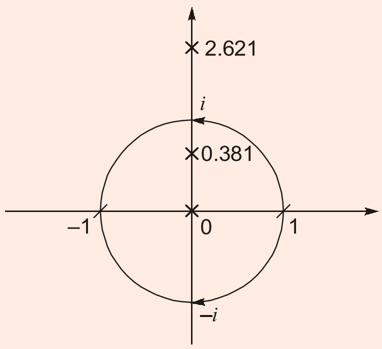
$$d\theta = \frac{dz}{z \cdot i}$$

Integral, $$I = \int_C \frac{\left(z - \frac{1}{z}\right)}{6i - 2\left(z - \frac{1}{z}\right)} \cdot \frac{dz}{zi}$$

$$I = \frac{1}{i} \int_C \frac{(z^2 - 1)}{6zi - 2(z^2 - 1)} \cdot \frac{dz}{z}$$

$$= \frac{1}{i} \int_C f(z)dz \text{ where } C \text{ is the unit circle } |z| = 1$$

$\because$ $$f(z) = \frac{-(z^2 - 1)}{z(2z^2 - 6zi - 2)}$$



Poles of $f(z)$ are $z = 0, 0.38i, 2.6zi$. So, here only two poles lies inside the unity circle.

Residues,

$$\text{Res}(0) = \frac{-1}{2}$$

$$\text{Res}(0.38i) = \frac{-(6.38i)^2 - 1}{0.38i(0.38i - 2.62i)}$$

$$= \frac{1.144}{0.38i(-2.24i)}$$

$$= \frac{1.144}{0.85}$$

$$\text{Res}(0.38i) = 1.344$$

$$I = \frac{1}{i}\left(2\pi i \times \left(-\frac{1}{2} + 1.344\right)\right)$$

$$= 2\pi \times 0.844$$

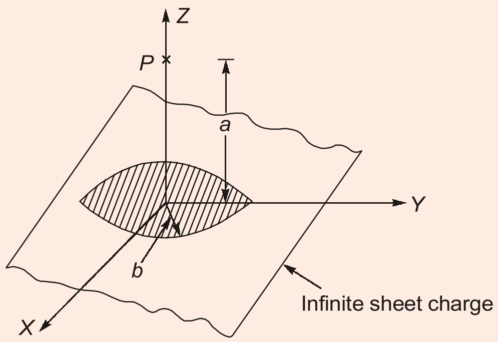
$$I = 5.303$$

End of Solution

Q.3 (b) What is the electric field intensity $\vec{E}_1$ V/m due to an infinite sheet of uniform charge density σ C/m^2?

(i) Derive the electric field intensity $\vec{E}_2$ V/m at *P* contributed by the circular portion of this infinite sheet charge of radius *b* metre on the perpendicular axis at *a* metre from the sheet as shown in the figure below :

(ii) Find *b*, if *a* = 0.5 m and $\vec{E}_2 = \frac{\vec{E}_1}{2}$.



[20 marks : 2023]

Solution:

From Gauss law,

$$\oint \bar{D} \cdot d\bar{s} = Q_{enc}$$

$$Q_{enc} = \int \sigma ds = \sigma \int ds = \sigma A$$

$$\oint \bar{D}\cdot d\bar{s} = D\left[\underbrace{\int ds}_{\text{top}} + \underbrace{\int ds}_{\text{bottom}}\right]$$

$$= D[A + A]$$

$$= D[2A]$$

$\therefore$ $$D2A = \sigma A$$

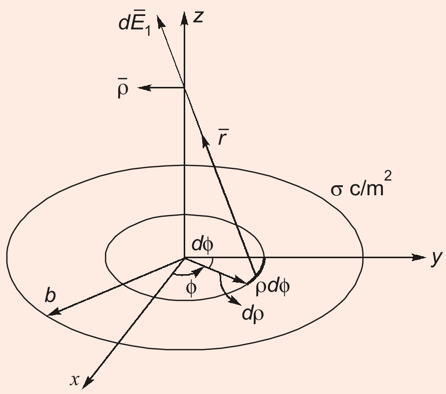
$\Rightarrow$ $$D = \frac{\sigma}{2}$$

In terms of vector, $$\bar{D} = \frac{\sigma}{2}\hat{a}_n$$

$\hat{a}_n$ = Unit normal vector from surface charge and taken towards point of interest.

$\Rightarrow$ $$\bar{E}_1 = \frac{\sigma}{2\epsilon_o}\hat{a}_n \text{ V/m}$$

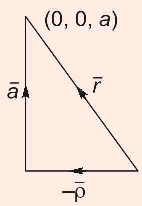
(i) The electric field is given as



$$d\bar{E}_2 = \frac{Kdq}{r^2}\hat{a}_r; \quad K = \frac{1}{4\pi\epsilon_o}$$

Here, $$dq = \sigma ds = \sigma\rho d\rho d\phi$$

$\hat{a}_r$:



$$\bar{r} = -\bar{\rho} + \bar{a}$$

$\Rightarrow$ $$\bar{r} = -\rho\hat{a}_\rho + a\hat{a}_z$$

$\Rightarrow$ $$|\bar{r}| = \sqrt{\rho^2 + a^2}$$

$\therefore \qquad \hat{a}_r = \frac{\bar{r}}{|\bar{r}|} = \frac{-\rho\hat{a}_\rho + a\hat{a}_z}{\sqrt{a^2+\rho^2}}$

Hence, $\qquad d\bar{E}_2 = \frac{K\sigma\rho d\rho d\phi}{(\rho^2+a^2)^{3/2}}(-\rho\hat{a}_\rho + a\hat{a}_z)$

The sum of the contribution along ρ gives zero due to symmetry.

Hence, $\qquad d\bar{E}_2 = \frac{K\sigma\rho d\rho d\phi a}{(\rho^2+a^2)^{3/2}}\hat{a}_z$

$\Rightarrow \qquad \bar{E}_2 = \int d\bar{E} = K\sigma a\int\frac{\rho d\rho}{(\rho^2+a^2)^{3/2}}\int d\phi\hat{a}_z$

Let $\qquad \rho^2 + a^2 = t \Rightarrow 2\rho d\rho = dt \Rightarrow \rho d\rho = \frac{dt}{2}$

At $\rho = 0$, $\qquad t = a^2$

At $\rho = b$, $\qquad t = b^2 + a^2$

$\therefore \qquad \bar{E}_2 = \frac{K\sigma a}{2}\int_{t=a^2}^{a^2+b^2}\frac{dt}{t^{3/2}}\int_{\phi=0}^{2\pi}d\phi\hat{a}_z = \frac{K\sigma a}{2}\times\left.\frac{-2}{\sqrt{t}}\right|_{a^2}^{a^2+b^2}\cdot 2\pi\hat{a}_z$

$\Rightarrow \qquad \bar{E}_2 = -\frac{K\sigma a}{2}\left[\frac{1}{\sqrt{a^2+b^2}} - \frac{1}{a}\right]2\pi\hat{a}_z$

$\Rightarrow \qquad \bar{E}_2 = \frac{1}{4\pi\epsilon_o}\sigma a 2\pi\left[\frac{1}{a} - \frac{1}{\sqrt{a^2+b^2}}\right]\hat{a}_z$

$\Rightarrow \qquad \bar{E}_2 = \frac{\sigma a}{2\epsilon_o}\cdot\frac{1}{a}\left[1 - \frac{a}{\sqrt{a^2+b^2}}\right]\hat{a}_z$

Hence, $\qquad \bar{E}_2 = \frac{\sigma}{2\epsilon_o}\left[1 - \frac{a}{\sqrt{a^2+b^2}}\right]\hat{a}_z$ V/m

(ii) Given : $\qquad \bar{E}_2 = \frac{\bar{E}_1}{2}$

$\Rightarrow \qquad \frac{\sigma}{2\epsilon_o}\left[1 - \frac{a}{\sqrt{a^2+b^2}}\right] = \frac{\sigma}{2\epsilon_o}\times\frac{1}{2}$

$\Rightarrow \qquad 1 - \frac{a}{\sqrt{a^2+b^2}} = \frac{1}{2}$

$\Rightarrow \qquad \frac{a}{\sqrt{a^2+b^2}} = \frac{1}{2}$

$\Rightarrow \qquad 2a = \sqrt{a^2+b^2}$

$\Rightarrow \qquad 4a^2 = a^2 + b^2$

$\Rightarrow \qquad 3a^2 = b^2$

$\Rightarrow \quad b = \sqrt{3}a = 0.5\sqrt{3}$

$\therefore \quad b = 0.5\sqrt{3}$ m

End of Solution

Q.3 (c) (i) In low-voltage Schering bridge designed for measurement of permittivity, the branch *ab* consists of two electrodes between which the specimen under test may be inserted; arm *bc* is a non-reactive resistor R_3 in parallel with a standard capacitor C_3 and *cd* is a non-reactive resistor R_4 in parallel with a standard capacitor C_4; arm *da* is a standard air capacitor of capacitance C_2. Without the specimen between the electrodes, balance is obtained with the following values :

$$C_3 = 150 \text{ pF}$$
$$C_4 = 200 \text{ pF}$$
$$C_2 = 250 \text{ pF}$$
$$R_3 = 5 \times 10^3 \ \Omega$$
$$R_4 = 10 \times 10^3 \ \Omega$$

With specimen inserted, these values become

$$C_3 = 200 \text{ pF}$$
$$C_4 = 1200 \text{ pF}$$
$$C_2 = 1000 \text{ pF}$$

and R_3 and R_4 remain as previous. In each case, the frequency is $\omega = 10 \times 10^3$ rad/s. Determine the relative permittivity of the specimen.

(ii) Draw the connections and phasor diagram of Anderson's bridge along with its advantages and disadvantages.

[10 + 10 marks : 2023]

Solution:

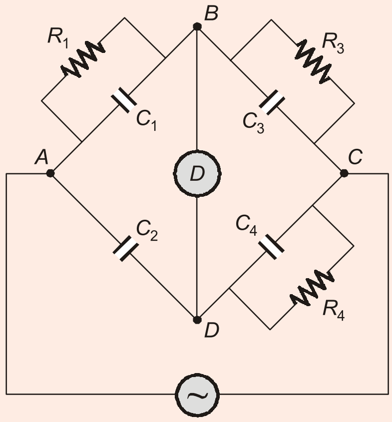
(i) For a Schering bridge, it is given that

Arm BC : Non-reactive resistor R_3 in parallel with a standard capacitor C_3.

Arm CD : Non-reactive resistor R_4 in parallel with a standard capacitor C_4.

Arm DA : Standard air capacitor of capacitance C_2.

With the above specified parameters, the low voltage schering bridge can be drawn as below :



Given : Without specimen between the electrodes of capacitor C_1, the balance is obtained with following values :

$$C_3 = 150 \text{ pF}, C_4 = 200 \text{ pF and } C_2 = 250 \text{ pF}$$
$$R_3 = 5 \times 10^3\ \Omega \text{ and } R_4 = 10 \times 10^3\ \Omega$$

With the specimen inserted,

$$C_3 = 200 \text{ pF}, C_4 = 1200 \text{ pF and } C_2 = 1000 \text{ pF}$$
$$R_3 = 5 \times 10^3\ \Omega \text{ and } R_4 = 10 \times 10^3\ \Omega$$

Here,

$$Y_1 = \text{Admittance of branch } AB = \frac{1}{R_1} + j\omega C_1$$
$$Y_2 = \text{Admittance of branch } AD = j\omega C_2$$
$$Y_3 = \text{Admittance of branch } BC = \frac{1}{R_3} + j\omega C_3$$
$$Y_4 = \text{Admittance of branch } CA = \frac{1}{R_4} + j\omega C_4$$

At balance condition,

$$Y_1 Y_4 = Y_2 Y_3$$
$$\left(\frac{1}{R_1} + j\omega C_1\right)\left(\frac{1}{R_4} + j\omega C_4\right) = j\omega C_2\left(\frac{1}{R_3} + j\omega C_3\right)$$
$$\left(\frac{1}{R_1 R_4} - \omega^2 C_1 C_2\right) + j\omega\left(\frac{C_1}{R_4} + \frac{C_4}{R_1}\right) = -\omega^2 C_2 C_3 + \frac{j\omega C_2}{C_3}$$

Separating real and imaginary parts and equating, we have

$$\frac{1}{R_1 R_4} - \omega^2 C_1 C_4 = -\omega^2 C_2 C_3 \qquad \text{...(i)}$$

and

$$\frac{C_4}{R_1} + \frac{C_1}{R_4} = \frac{C_2}{R_3}$$
$$C_1 = \frac{C_2 R_4}{R_3} - \frac{C_4 R_4}{R_1} \qquad \text{...(ii)}$$

From equation (i)

$$\omega^2[-C_2C_3 + C_1C_4] = \frac{1}{R_1R_4}$$

$$R_1 = \frac{-1}{\omega^2 R_4(C_2C_3 - C_1C_4)}$$

Substituting in eqn. (ii), we get

$$C_1 = \frac{C_2R_4}{R_3} + \omega^2 C_4 R_4^2 (C_2C_3 - C_1C_4)$$

$$C_1[1+\omega^2 C_4^2 R_4^2] = \frac{C_2R_4}{R_3} + \omega^2 C_2C_3C_4R_4^2$$

$$C_1 = \frac{\frac{C_2R_4}{R_3} + \omega^2 C_2C_3C_4R_4^2}{1+\omega^2 C_4^2 R_4^2}$$

Here, $\omega^2 C_2C_3C_4R_4^2 \ll \frac{C_2R_4}{R_3}$

and $\omega^2 C_4^2 R_4^2 \ll 1$

Hence, $C_1 = \frac{C_2R_4}{R_3}$

When there is no dielectric in the capacitor C_1, let the capacitance be C_0, we have

$$C_0 = \frac{C_2R_4}{R_3}$$

$$= \frac{250\text{ pF} \times 10 \times 10^3}{5 \times 10^3} = 500\text{ pF} \quad \text{...(iii)}$$

When the specimen is inserted as dielectric, let its capacitance be C_s, we have

$$C_s = \frac{C_2R_4}{R_3} = \frac{1000\text{ pF} \times 10 \times 10^3}{5 \times 10^3}$$

$$= 2000\text{ pF} \quad \text{...(iv)}$$

Without specimen between electrodes, the capacitance is given by

$$C_0 = \frac{\varepsilon_o A}{D} \quad (\varepsilon_r = 1 \text{ for free space})$$

With specimen between electrodes, the capacitance is given by

$$C_s = \frac{\varepsilon_o \varepsilon_r A}{d}$$

where, ε_r = relative permittivity of the specimen

From above, $\frac{C_s}{C_o} = \varepsilon_r$

Using values of C_o and C_s from equations (iii) and (iv)

$$\varepsilon_r = \frac{C_s}{C_o} = \frac{2000}{500} = 4$$

Hence, the relative permittivity of the specimen is 4.

(ii) **Anderson's Bridge :** This bridge, in fact, is a modification of the Maxwell's inductance capacitance bridge. In this method, the self-inductance is measured in terms of a standard capacitor. This method is applicable for precise measurement of self-inductance over a very wide range of values.

Figure shows the connections and the phasor diagram of the bridge for balanced conditions.

Let L_1 = self-inductance to be measured,

R_1 = resistance of self-inductor,

r_1 = resistance connected in series with self-inductor,

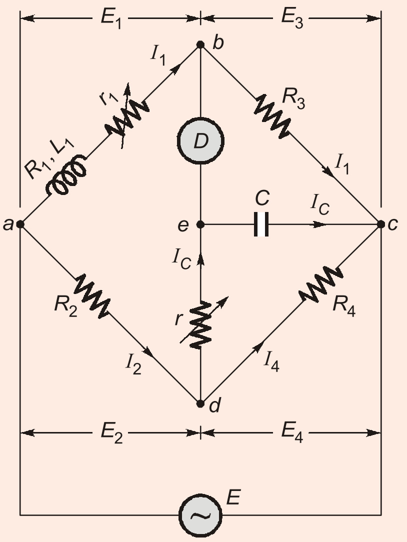
r, R_2, R_3, R_4 = known non-inductive resistances,

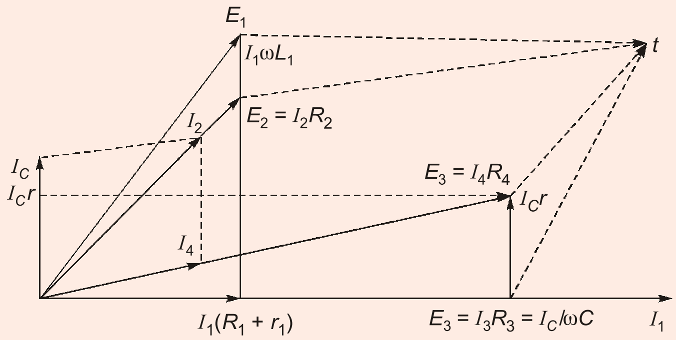
and C = fixed standard capacitor

At balance, $I_1 = I_3$ and $I_2 = I_c + I_4$

Now, $$I_1 R_3 = I_C \times \frac{1}{j\omega C}$$

$\therefore$ $$I_c = I_1 j\omega C R_3$$





Writing the other balance equations

$$I_1(r_1 + R_1 + j\omega L_1) = I_2 R_2 + I_c r$$

and $$I_c\left(r + \frac{1}{j\omega C}\right) = (I_2 - I_c)R_4$$

Substituting the value of I_c in the above equation, we have

$$I_1(r_1 + R_1 + j\omega L_1) = I_2 R_2 + I_1 j\omega C R_3 r \text{ or } I_1(r + R_1 + j\omega L_1 - j\omega C R_3 r)$$
$$= I_2 R_2 \quad \text{...(i)}$$

and $$j\omega C R_3 I_1\left(r + \frac{1}{j\omega C}\right) = (I_2 - I_1 j\omega C R_3)R_4$$

or $$I_1(j\omega C R_3 r + j\omega C R_3 R_4 + R_3) = I_2 R_4 \quad \text{...(ii)}$$

From eqn. (i) and (ii), we obtain

$$I_1(r_1 + R_1 + j\omega L_1 - j\omega CR_3 r) = I_1\left(\frac{R_2R_3}{R_4} + \frac{j\omega CR_2R_3r}{R_4} + j\omega CR_3R_2\right)$$

Equating the real and the imaginary parts :

$$R_1 = \frac{R_2R_3}{R_4} - r_1$$

and
$$L_1 = C\frac{R_3}{R_4}[r(R_4 + R_2) + R_2R_4]$$

An examination of balance equations reveals that to obtain easy convergence of balance, alternate adjustments of r_1 and r should be done as they appear in only one of the two balance equations.

Advantages :

1. In case adjustments are carried out by manipulating control over r_1 and r_2 they become independent of each other. This is a marked superiority over sliding balance conditions met with low Q coils when measuring with Maxwell's bridge. A study of convergence conditions would reveal that it is much easier to obtain balance in the case of Anderson's bridge than in Maxwell's bridge for low Q-coils.
2. A fixed capacitor can be used instead of a variable capacitor as in the case of Maxwell's bridge.
3. This bridge may be used for accurate determination of capacitance in terms of induction.

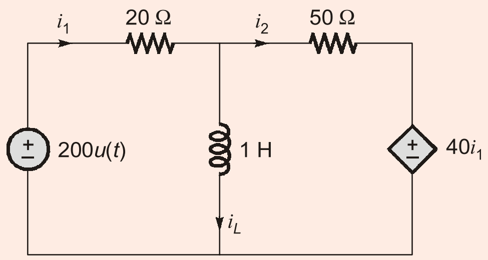
Disadvantages :

1. The Anderson's bridge is more complex than its prototype Maxwell's bridge. The Anderson's bridge has more parts and is more complicated to set up and manipulate. The balance equations are not simple and in fact are much more tedious.
2. An additional junction point increases the difficulty of shielding the bridge.

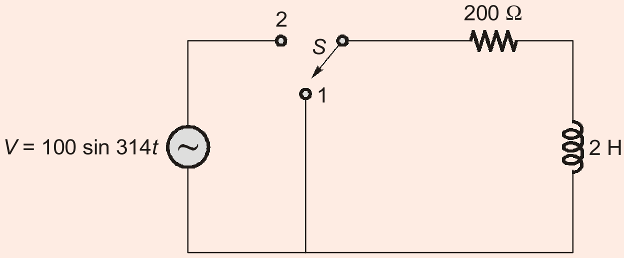
Considering the above complications of the Anderson's bridge, in all the cases where a variable capacitor is permissible the more simple Maxwell's bridge is used instead of Anderson's bridge.

End of Solution

Q.4 (a) (i) For the network shown in the following figure, compute $i_L(t)$ and $i_1(t)$, if the initial current through the inductor is 0 ampere :



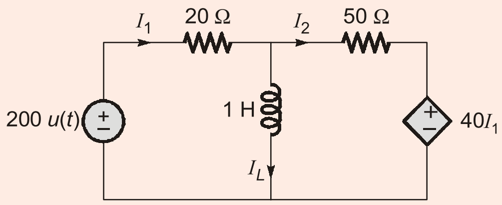
(ii) Determine the current at $t > 0$, if AC voltage V is applied, when switch S is moved from position 2 to position 1 at t = 0, for the network shown in the following figure. Assume steady-state current of 1 ampere in the network when the switch is at position 1 :



[10 + 10 marks : 2023]

Solution:

(i) For the network shown in the figure. Find $i_L(t)$ and $i_1(t)$. If initial current in the inductor is zero,



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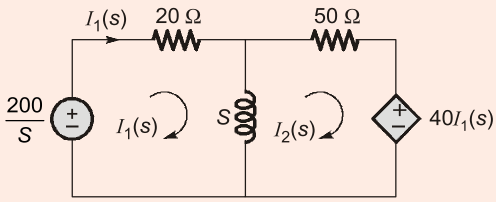


$$\frac{200}{s} = (20 + s)\, I_1(s) - sI_2(s) \qquad \text{...(i)}$$

$$(50 + S)\, I_2(s) - sI_1(s) + 40I_1(s) = 0 \qquad \text{...(ii)}$$

from equation (ii),

$$I_2(s) = \frac{(s-40)I_1(s)}{(50+s)} \qquad \text{...(iii)}$$

Substitute equation (iii) in equation (i),

$$\frac{200}{s} = (20+s)I_1(s) - \frac{s(s-40)}{(s+50)}I_1(s)$$

$$\frac{(s+50)200}{s} = (20 + s - s^2 + 40s)\, I_1(s)$$

$$I_1(s) = \frac{200s + 10^4}{110s^2 + 1000s} \qquad \text{...(iv)}$$

$$i_1(t) = L^{-1}\, I_1(s)$$

$$= \left[10 - \frac{90}{11} e^{\frac{-100}{11}t}\right]$$

$$I_L(s) = I_1(s) - I_2(s)$$

$$= I_1(s) - \frac{(s-40)}{(50+s)} I_1(s)$$

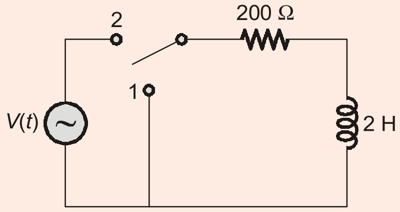
$$= \left(\frac{s+50-s+40}{s+50}\right) I_1(s)$$

$$I_L(s) = \frac{18000}{s[110s+1000]} = \frac{1800}{s[11s+1000]}$$

$$i_L(t) = L^{-1} I_L(s)$$

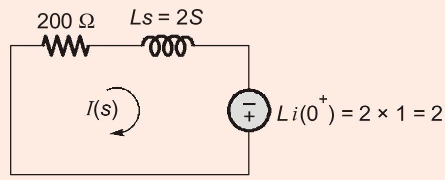
$$= 18 - 18e^{\frac{-100}{11}t}$$

(ii) Determine the current at $t > 0$, when AC voltage is applied, when switch is moved from position-2 to position-1 at $t = 0$ sec. Assume steady state current of 1 A in the network, when the switch is at position-1.



$$V(t) = 100 \sin 314t$$

At $t > 0$



$$(200 + 2s) I(s) = 2$$

$$I(s) = \frac{2}{200+2s}$$

$$i(t) = L^{-1} I(s)$$

$$i(t) = [e^{-100t}] u(t)$$

End of Solution

Q.4 (b) (i) Give the variation of resistivity of purified mercury with temperature. Also, represent the resistivity of normal metal as a function of temperature (T) along with pure and impure superconductors.

(ii) Compute the drift mobility and mean scattering time of conduction electrons in copper at room temperature, given that the density of copper is

8.98 g/cm^3, the conductivity of copper is $5.95 \times 10^5\ \Omega^{-1}\ \text{cm}^{-1}$ and the atomic mass of copper is 63.5 g/mol. Take Avogadro number, $N_A = 6.02 \times 10^{23}$ and charge on electron (e) = 1.6×10^{-19} coulomb, mass of electron (m_e) = 9.1×10^{-31} kg.

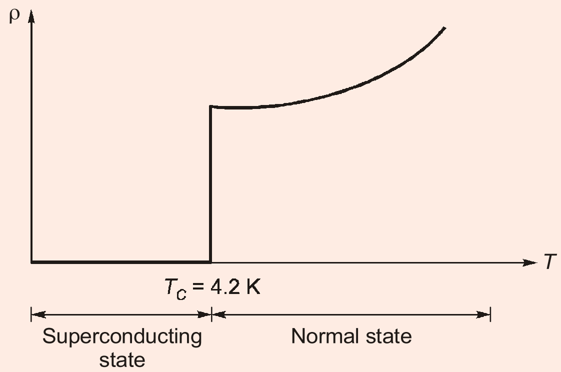
[12 + 8 marks : 2023]

Solution:

(i) Below the critical temperature (T_c = 4.2 K), mercury enters into superconducting state and its resistivity becomes zero.

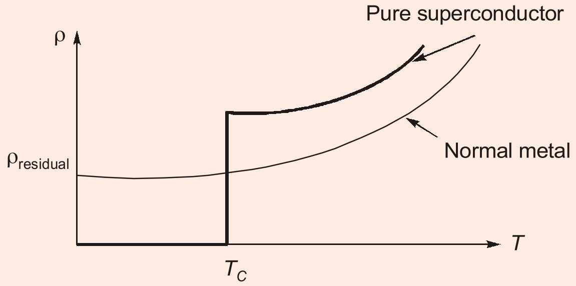
For temperature $T > T_c$, it behaves like normal conductor.

Following figure shows the resistivity (ρ) of mercury with temperature (T).

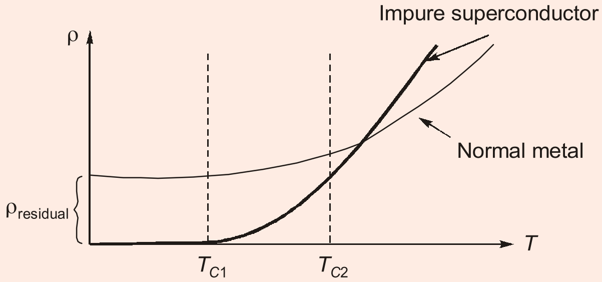


There is an abrupt change in resistivity of mercury at critical temperature (T_c).

Pure superconductors are type-I superconductors and following figure represents comparison of resistivity of pure superconductor and normal metal.



Impure superconductors are type-II superconductors and following figure represents comparison with normal metal.



(ii) Given :

$$\rho_{Cu} = 8.98 \text{ g/cm}^3$$
$$\sigma = 5.95 \times 105\ \Omega^{-1}\ \text{m}^{-1}$$

$$A_{Cu} = 63.5 \text{ g/mol}$$
$$N_A = 6.02 \times 10^{23} \text{ atoms/mol}$$

Atomic concentration in Cu

$$\rho_{Cu} = \frac{n_A}{V_C} \cdot \frac{A_{Cu}}{N_A} \quad \Rightarrow \quad \frac{n_A}{V_C} = \frac{\rho_{Cu} \times N_A}{A_{Cu}} = N$$

$$N = \frac{8.98 \text{ g/cm}^3 \times 6.02 \times 10^{23} \text{ atoms/mol}}{63.5 \text{ g/mol}}$$

$$N = 8.51 \times 1022 \text{ atoms/cm}^3$$

Each copper atom contributes 2 electrons [i.e., Cu^{2+} ion].
Hence, electron concentration in Cu,

$$n = 2N = 17.02 \times 10^{22} \text{ electrons/cm}^3$$

$$\Rightarrow \quad n = \frac{17.02 \times 10^{22}}{(10^{-2})^3 \text{m}^3} = 1.702 \times 10^{29} \text{ electrons/m}$$

Now, conductivity

$$\sigma = nq\mu_n \quad \Rightarrow \quad \mu_n = \frac{\sigma}{nq}$$

$$\Rightarrow \quad \mu_n = \frac{5.95 \times 10^5}{1.702 \times 10^{29} \times 1.6 \times 10^{-19}}$$
$$= 2.185 \times 10^{-5} \text{ m}^2/\text{V}$$

$$\Rightarrow \quad \mu_n = 2.185 \times 10^{-5} \text{ m}^2/\text{V-s} = \text{mobility}$$

$$\mu_n = \frac{q\tau_C}{m_n} \quad \Rightarrow \quad \tau_C = \frac{\mu_n m_n}{q} = \text{mean scattering time}$$

$$\Rightarrow \quad \tau_c = \frac{2.185 \times 10^{-5} \times 9.1 \times 10^{-31}}{1.6 \times 10^{-19}}$$

$$\Rightarrow \quad \tau_c = 1.25 \times 10^{-16} \text{ s}$$

End of Solution

Q.4 (c) (i) A moving-coil instrument has a resistance of 5 Ω between terminals and full-scale deflection is obtained with a current of 15 mA. This instrument is to be used with a manganin shunt to measure 100 A at full scale. Calculate the error caused by a 10°C rise in temperature :

(1) when the internal resistance of 5 Ω is due to copper only;

(2) when a 3 Ω manganin swamping resistance is used in series with a copper coil of 2 Ω resistance.

(The resistance temperature coefficients of copper and manganin are 0.004/°C and 0.000015/°C respectively).

(ii) Draw the block diagram of ramp-type digital voltmeter and explain its functioning.

[10 + 10 marks : 2023]

Solution:

(i) Given that :

$$I_m = 15 \text{ mA}, I = 100 \text{ A}$$

$$m = \frac{I}{I_m} = \frac{100}{15\times10^{-3}} = 6666.67$$

$$R_{sh} = \frac{R_m}{(m-1)} = \frac{5\ \Omega}{(6666.67-1)} = 0.75 \text{ m}\Omega$$

Meter resistance for 10°C rise in temperature.

$$R_{mt} = 5(1 + 0.004 \times 10) = 5.2\ \Omega$$

Shunt resistance for 10°C rise in temperature

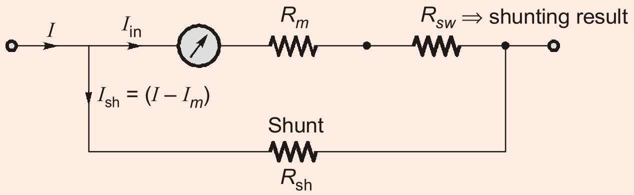
$$R_{sh} = 0.75 \text{ m}\Omega(1 + 0.000015 \times 10) = 0.7501125 \text{ m}\Omega$$

Current I_{mt} through the meter for 100 A in the main circuit for 10°C rise in temperature

$$I_{mt} = 100\times\frac{0.7501125 \text{ m}\Omega}{5.2+0.7501125 \text{ m}\Omega}$$

$$= 0.014423 \text{ Amp} = 14.423 \text{ mA}$$

$$\% \text{ Errror} = \frac{14.423 \text{ mA} - 15 \text{ mA}}{15 \text{ mA}}\times100 = -3.845\%$$



Total resistance in the meter circuit,

$$R_m + R_{SW} = 2 + 3 = 5\ \Omega$$

$$R_{sh} = \frac{R_m}{(m-1)} = \frac{5\ \Omega}{6666.67-1} = 0.75 \text{ m}\Omega$$

Resistance of the meter for 10°C rise in temperature

$$R_{mt} = 2\ \Omega[1 + 0.004 \times 10] + 3\ \Omega(1 + 0.000015 \times 10]$$

$$R_{mt} = 2[1 + 0.04] + 3[1 + 0.00015] = 5.08045\ \Omega$$

Shunt resistance for 10°C rise in temperkature

$$R_{sht} = 0.75 \text{ m}\Omega[1 + 0.000015 \times 10] = 0.7501125 \text{ m}\Omega$$

$$I_{mt} = 100\times\frac{0.7501125 \text{ m}\Omega}{5.08045+0.7501125 \text{ m}\Omega}$$

$$= 0.0147625 \text{ Amp}$$

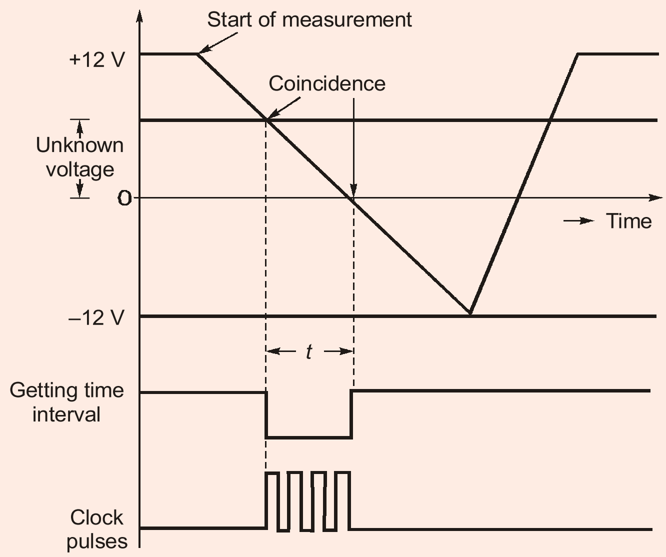
$$I_{mt} = 14.7625 \text{ Amp}$$

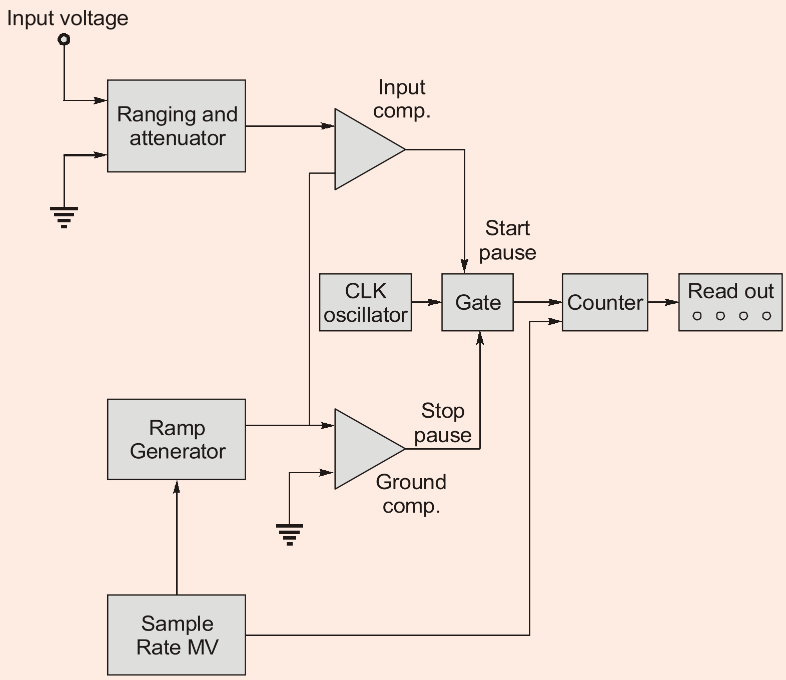
$$\% \text{ Error} = \frac{14.7625-15}{15}\times100$$

$$= -1.5833\%$$

(ii) **Ramp Type Digital Voltmeter :** The operating principle of a ramp type digital voltmeter is to measure the time that a linear ramp voltage takes to change from level of input to zero voltage (or vice versa). This time interval is measured with an electronic time interval counter and the count is displayed as a number of digits on electronic indicating tubes of the output readout of the voltmeter.

The conversion of a voltage value of a time interval is shown in the timing diagram of figure.





At the start of measurement a ramp voltage is initiated. A negative going ramp is shown in figure but a positive going ramp may also be used. The ramp voltage value is continuously compared with the voltage being measured (unknown voltage). At the instant the value of ramp voltage is equal to that of unknown voltage a coincidence circuit, called an input comparator, generates a pulse which opens a gate (see figure). The ramp voltage continues to decrease till it reaches ground level (zero voltage). At this instant another comparator called ground comparator generates a pulse and closes the gate.

The time elapsed between opening and closing the gate is *t* as indicated in figure. During this time interval pulses from a clock pulse generator pass through the gate and are counted and displayed.

The decimal number as indicated by the readout is a measure of the value of input voltage.

The sample rate multivibrator determines the rate at which the measurement cycles are initiated. The sample rate circuit provides an initiating pulse for the ramp generator to start its next ramp voltage. At the same time it sends a pulse to the counters which sets all of them to 0. This momentarily removes the digital display of the readout.

End of Solution

SECTION : B

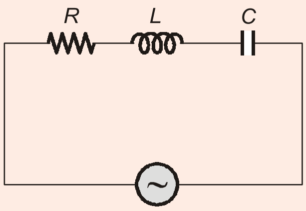
Q.5 (a) For a series R-L-C circuit excited from an AC source, find the resonant frequency, bandwidth and quality factor, if $R = 100\ \Omega$, $L = 0.5$ H and $C = 0.4\ \mu$F.

[12 marks : 2023]

Solution:

For series RLC circuit, excited from AC source. The resonant frequency, bandwidth qaulity factor when,

$$R = 100\ \Omega$$
$$L = 0.5\ \text{H}$$
$$C = 0.4\ \mu\text{F}$$



1. $$\omega_0 = \frac{1}{\sqrt{LC}} = 2.23 \times 10^3 \text{ rad/s}$$

2. $$\text{B.W.} = \omega_2 - \omega_1 = \frac{R}{L} = 200 \text{ rad/s}$$

3. $$Q = \frac{\omega_0}{\text{B.W.}} = \frac{\omega_0}{\omega_2 - \omega_1} = 11.15$$

End of Solution

Q.5 (b) Define line defects in materials. Explain different types of line defects and compare them. Also, explain their cause of creation.

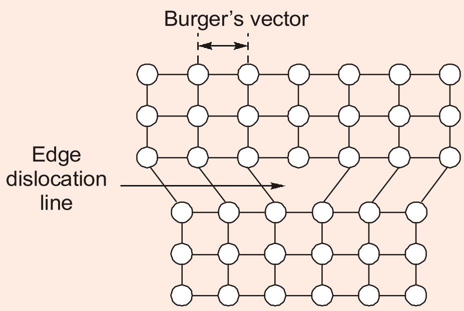
[12 marks : 2023]

Solution:

Line defect is the defect confined to more number of atoms in a lattice.

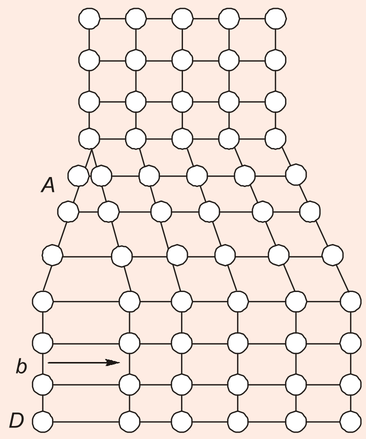
Linear defect: A dislocation is a linear or one dimensional defect around which some of the atoms are misaligned. There are following types of linear dislocation as given below:

1. **Edge dislocation:** In an edge dislocation, an extra portion of a plane of atoms, or half-plane appears and the edge of which terminates within the crystal. It is a linear defect that centers around the line that is defined along the end of the extra half plane of atoms. This is sometimes termed as dislocation line, which for the edge dislocation as shown in figure, is perpendicular to the plane of the page. Within the region around the dislocation line there is some localized lattice distortion. The atoms above the dislocation line are squeezed together and those below are pulled apart. This is reflected in the slight curvature for the vertical planes of atoms as they bend around this extra half plane. The magnitude of this dislocation decreases with distance away from the dislocation line. An edge dislocation may also be formed by an extra half plane of atoms that is included in the bottom portion of the crystal.



Edge dislocation

2. **Screw dislocation:** Screw dislocation is formed by shear stress that is applied to produce the distortion. The upper front region of the crystal is shifted one atomic distance to the right relative to the bottom portion. The atomic distortion associated with a screw dislocation is also linear and along a dislocation line. The screw dislocation derivers its name from the spiral or helical path or ramp that is traced around the dislocation line by the atomic planes of atoms.



Screw dislocation

3. **Mixed dislocation:** Most dislocations found in crystalline materials are probably neither pure edge nor pure screw, but exhibit components of both types, these are termed as mixed dislocations.

The nature of a dislocation is defined by the relative orientations of dislocation line and Burger's vector. For an edge, they are perpendicular, whereas for a screw, they are parallel. They are neither parallel nor perpendicular for a mixed dislocation. Also, even though a dislocation changes direction and nature within a crystal (e.g. from edge to mixed to screw), the Burger's vector will be the same at all points along its line.

End of Solution

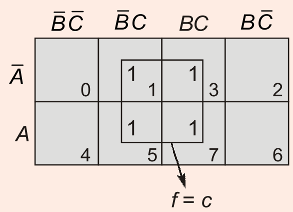
Q.5 (c) Design a circuit that accepts a 3-bit number and gives an output 0, if input represents even decimal number and gives an output 1, if input represents an odd decimal number.

[12 marks : 2023]

Solution:

Truth Table :

	A	B	C	f
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	1



End of Solution

Q.5 (d) A current transformer having a bar primary is rated at 500/5 A, 50 Hz with an output of 20 VA. At rated load with non-inductive burden, the inphase and quadrature components (referred to the flux) of the exciting mmf are 8 A and 10 A respectively. The number of turns in the secondary winding is 98 and the impedance of the secondary winding is (0.4 + *j*0.3) Ω. Calculate the ratio and phase angle errors.

[12 marks : 2023]

Solution:

Primary winding turns, $N_P = 1$

Secondary winding turns, $N_S = 98$

∴ Turns ratio, $n = 98$

Nominal ratio, $K_n = \dfrac{500}{5} = 100$

Magnetizing current components is in phase while the loss component is in quadrature with the flux.

∴ Magnetizing mmf = 8 A

Mmf equivalent ot loss = 10 A

Magnetizing current, $I_m = \dfrac{\text{Magnetizing mmf}}{\text{Primary winding turns}}$

$= \dfrac{8}{1} = 8 \text{ A}$

Loss component, $I_e = \dfrac{\text{Loss mmf}}{\text{Primary winding turns}}$

$= \dfrac{10}{1} = 10 \text{ A}$

Output volt-ampere, $VA = 20$

Impedance of secondary load burden

$= \dfrac{20}{(5)^2} = \dfrac{20}{25} = 0.8\ \Omega$

It is given that the external burden is purely resistive.

∴ Resistance of external burden = 0.8 Ω. Reactance of external burden = 0 Ω.

Resistance of total secondary circuit burden

= 0.8 Ω + 0.4 Ω

= 1.2 Ω

Reactance of total secondary circuit burden

= 0 + 0.3 Ω

= 0.3 Ω

Secondary phase angle,

$\delta = \tan^{-1}\dfrac{0.3}{1.2}$

$= 14.036°$

So, $\cos\delta = 0.970$ and $\sin\delta = 0.242$

Actual ratio, $$R = \frac{I_e\cos\delta + I_m\sin\delta}{I_s} + n$$

$$= 98 + \frac{10\times1.2 + 8\times0.3}{5}$$

$$= 98 + \frac{12+2.4}{5}$$

$$= 100.88$$

Ratio (current) error $$= \frac{100-100.88}{100.88}\times100$$

$$= -0.87\%$$

Phase angle, $$\theta = \frac{180}{\pi}\left[\frac{I_m\cos\delta - I_e\sin\delta}{nI_s}\right]$$

$$= \frac{180}{\pi}\left[\frac{8\times1.2 - 10\times0.3}{98\times5}\right]$$

$$= 0.771^\circ$$

End of Solution

Q.5 (e) (i) The reverse recovery time t_{rr} of a diode is 2 μs. In the conducting mode to reverse blocking mode operation, the diode needs the rate of fall of forward current of 50 amperes/μs. Determine the stroage charge and the peak reverse current.

(ii) A diode with 500 mW power dissipation at 25°C has 5 mW/°C derating factor. If the forward voltage drop remains constant at 0.7 V, calculate the maximum forward current at 50°C.

[6 + 6 marks : 2023]

Solution:

(i) $t_{rr} = 2\ \mu s$

$$\frac{di}{dt} = \frac{I_F}{T} = 50\ \text{amp}/\mu s$$

Storage charge, $Q_S = ?$

Peak reverse current, $I_{RR} = ?$

$$Q_S = \frac{1}{2}\times t_{rr}^2\frac{di}{dt}$$

$$= \frac{1}{2}\times(2\times10^{-6})^2\times\frac{50\ \text{A}}{1\times10^{-6}}$$

$$Q_S = \frac{1}{2}\times2\times10^{-6}\times50$$

$$Q_S = 50\ \mu C$$

Peak reverse current I_{RR}

$$Q_S = \frac{1}{2} \times I_{RR} \times t_{rr}$$

$$I_{RR} = \frac{Q_S \times 2}{t_{rr}}$$

$$= \frac{50\mu C \times 2}{2\ \mu S}$$

$$I_{RR} = 50\ A$$

(ii) At 25° C, derating factor

$$D = 5\ mW/°C$$

$$V_D = 0.7\ V$$

I_F at 50° C = ?

$$\Delta P_D = D(T_j - T_A)$$

$$= 5\ mW/°C\ (50° - 25°\ C)$$

$$= 125\ mW$$

$$P_D(50°\ C) = 500\ mW - 125\ mW = 375\ mW$$

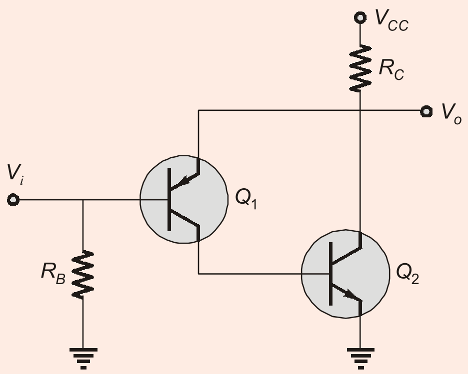
$$P_D(50°\ C) = V_D I_F$$

$$\Rightarrow \quad I_F = \frac{P_D}{V_D} = \frac{375\ mW}{0.7\ V}$$

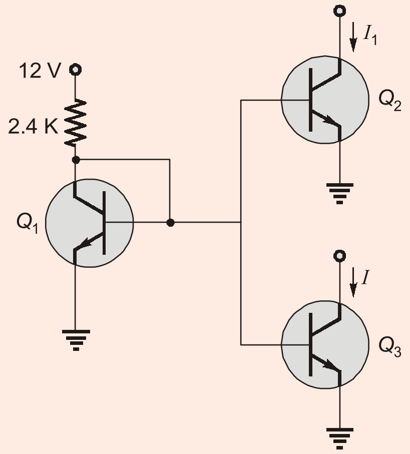
$$I_F = 535.71\ mA$$

End of Solution

Q.6 (a) (i) Calculate the DC voltages V_o and V_i, and the bias currents for the feedback pair of transistors shown in the figure below, given that $\beta_1 = 100$, $\beta_2 = 150$, $V_{CC} = 15$ volts, $R_C = 200\ \Omega$ and $R_B = 1\ M\Omega$:



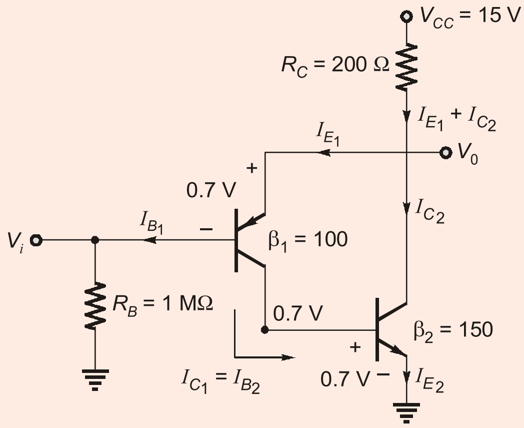
(ii) In the current mirror circuit shown in the figure below, the current is mirrored in two transistors. All the three transistors are identical. Calculate the load current I assuming $\beta = 100$:



[10 + 10 marks : 2023]

Solution:

(i)



$$V_o = 0.7 + V_{in}$$

$$V_{in} = I_{B1} \times R_B$$

$$= 10^6 I_{B1}$$

$$= 10^6 \frac{I_{E1}}{(1+\beta_1)}$$

$$= 10^6 \times \frac{I_{E1}}{100}$$

$$= 10^4 I_{E1} \quad \Rightarrow \quad V_o = 0.7 + 10^4 I_{E1} \quad \text{...(1)}$$

$$V_o = V_{CC} - (I_{E1} + I_{C2})R_C$$

$$= V_{CC} - (I_{E1} + \beta_2 I_{B2})R_C$$

$$= V_{CC} - (I_{E1} + \beta_2 I_{C1})R_C$$

$$= V_{CC} - (I_{E1} + \beta_2 I_{E1})R_C$$

where, $I_{C1} = I_{E1}$ (β is large)

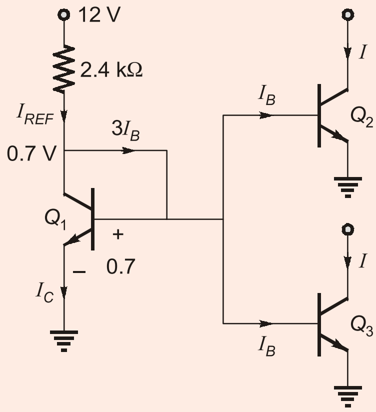
$$= V_{CC} - (1 + \beta_2)I_{E1}R_C \quad (\beta_2 >> 1)$$

$$= V_{CC} - 150 I_{E1} \times 200\ \Omega$$
$$= 15 - 3I_{E1} \times 10^4 \quad ...(2)$$

Eqn. (1) = (2)

$$0.7 + 10^4 I_{E1} = 15 - 3I_{E1} \times 10^4$$
$$4I_{E1} \times 10^4 = 15 - 0.7$$
$$4I_{E1} \times 10^4 = 14.3$$
$$I_{E1} = \frac{14.3}{4 \times 10^4} = 0.35\ \text{mA}$$
$$V_i = 10^4 I_{E1}$$
$$= 10^4 \times 0.35\ \text{mA}$$
$$= 10 \times 0.35 = 3.5\ \text{V}$$
$$V_i = 3.5\ \text{V}$$
$$V_o = 15 - 3I_{E1} \times 10^4$$
$$= 15 - 0.3 \times 0.35\ \text{mA} \times 10^4$$
$$= 15 - 1.05 \times 10$$
$$= 15 - 10.5 = 4.5\ \text{V}$$
$$V_o = 4.5\ \text{V}$$

(ii)



Q_1, Q_2, Q_3 are identical.

$$I_{REF} \text{ in } Q_1 = \frac{12 - 0.7}{2.4K}$$
$$= 4.7\ \text{mA}$$
$$I_{REF} = I_C + 3I_B$$
$$= I_C + 3\frac{I_C}{\beta}$$
$$= I_C\left(1 + \frac{3}{\beta}\right)$$

Here, $I_C = I$ (current mirror)

$$= I\left(1 + \frac{3}{\beta}\right)$$

$$= I_C + \frac{1}{\beta} = I_C\left(1+\frac{3}{\beta}\right)$$

Here, $I_C = I$ (current mirror)

$$= I\left(1+\frac{3}{\beta}\right)$$

$$I = \frac{I_{REF}}{1+\frac{3}{\beta}} = \frac{4.7 \text{ mA}}{1+\frac{3}{100}}$$

$$= \frac{4.7 \text{ mA}}{1.03} = 4.56 \text{ mA}$$

$$I = 4.56 \text{ mA}$$

End of Solution

Q.6 (b) State Gauss divergence theorem. Let R be the region bounded by the closed cylinder $x^2 + y^2 = 4$, $z = 0$ and $z = 2$. Verify this theorem, if $\vec{F} = 3x^2\hat{i} + y^2\hat{j} + z\hat{k}$.

[20 marks : 2023]

Solution:

Divergence Theorem : If $\bar{F}$ is differentiable vector point function in a closed surface S having volume V, then

$$\iint_S \bar{F}\cdot\hat{n}dS = \iiint_V \bar{\nabla}\cdot\bar{F}\,dV$$

Verify : We need to prove that

$$\iint_S \bar{F}\cdot\hat{n}ds = \iiint \bar{\nabla}\cdot\bar{F}dV$$

Consider LHS :

$$\iint_S \bar{F}\cdot\hat{n}ds = \iint_{S_1} \bar{F}\cdot\hat{n}ds + \iint_{S_2} \bar{F}\cdot\hat{n}ds + \iint_{S_3} \bar{F}\cdot\hat{n}ds$$

For S_1 : $Z = 0,\ \hat{n} = -\hat{k}$

$$\iint_{S_1} \bar{F}\cdot\hat{n}ds = \iint_{S_1} (-z)ds = 0 \qquad (\because z = 0)$$

For S_2 : $Z = 2,\ \hat{n} = \hat{k}$

$$\iint_{S_2} \bar{F}\cdot\hat{n}ds = \iint_{S_2} z\cdot ds = 2 \text{ (Area of circle)}$$

$$= 2(\pi \times 2^2)$$

$$= 8\pi$$

For S_3 : It is curved surface of cylinder. Project the surface onto XZ-plane.

$$\iint_{S_3} \bar{F}\cdot\hat{n}ds = \iint_R \bar{F}\cdot\hat{n}\cdot\frac{dxdz}{\hat{n}\cdot\hat{j}}$$

$$\hat{n} = \frac{xi + yj}{2}$$

$$\bar{F}\cdot\hat{n} = (3x^2 i + y^2 j + zk)\cdot\frac{(xi+yj)}{2} = \frac{3x^3 + y^3}{2}$$

and $$\hat{n}\cdot\hat{j} = \frac{y}{2}$$

$$\iint_{S_3} \bar{F}\cdot\hat{n}ds = \iint_R \frac{3x^3+y^3}{2}\cdot\frac{dxdz}{\frac{y}{2}}$$

$$= \iint_R \left(3\frac{x^3}{y} + y^2\right)dxdz$$

Put $y = 2\sin\theta$, $x = \cos\theta$, $dx = -2\sin\theta d\theta$

$$= \int_{z=0}^{2}\int_{\theta=0}^{2\pi}\left(3\left(\frac{8\cos^3\theta}{2\sin\theta}\right) + 4\sin^2\theta\right)(-2\sin\theta)d\theta dz$$

$$= \int_{z=0}^{2}\int_{0}^{2\pi}\left(-24\cos^3\theta - 8\sin^3\theta\right)d\theta dz$$

$$= \int_{z=0}^{2}\left(-24\int_0^{2\pi}\cos^3\theta d\theta\right) - \left(8\int_0^{2\pi}\sin^3\theta d\theta\right)dz$$

$$= \int_{z=0}^{2}\left[(-24)\times 2\int_0^{\pi}\cos^3\theta d\theta - 0\right]dz$$

$$= -48 \times 0 = 0$$

$\therefore$ $$\iint_S \bar{F}\cdot\hat{n}ds = \iint_{S_1}\bar{F}\cdot\hat{n}ds + \iint_{S_2}\bar{F}\cdot\hat{n}ds + \iint_{S_3}\bar{F}\cdot\hat{n}ds$$

$$= 0 + 8\pi + 0$$

$$= 8\pi$$

Consider RHS :

$$\iiint \bar{\nabla}\cdot\bar{F}dV = \iint_R \int_{z=0}^{2}(6x + 2y + 1)dzdydx$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(6x+2y+1)(z)_0^2\,dy\,dx$$

$$= 2\int_{-2}^{2}\left](6x+1)y + 0\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dx$$

$$= 2\times 2\int_{-2}^{2}(6x+1)\sqrt{4-x^2}\,dx$$

$$= 4\left[\int_{-2}^{2} 6x\sqrt{4-x^2}\,dx + \int_{-2}^{2} \sqrt{4-x^2}\,dx\right]$$

$$= 4\left[0 + 2\int_{-2}^{2} \sqrt{4-x^2}\,dx\right] = 4\left[2 \times \frac{1}{4}\pi(2)^2\right] = 8\pi$$

L.H.S. = R.H.S.

Hence verified.

End of Solution

Q.6 (c) (i) What is Random Access Memory? Explain Static Random Access Memory and Dynamic Random Access Memory.

(ii) Write a program in C to find whether the given number is even or odd and if it is odd, find whether it is prime or not.

[10 + 10 marks : 2023]

Solution:

(i) RAM (Random Access Memory) :

- The information stored in the RAM is lost when the power supply to the PC (or) laptop is switched off.
- RAM is a volatile memory. It is generally known as main memory.
- RAM is used to read and write the data into it which is accessed by the CPU randomly.
- RAM is used to store the data that is currently processed by the CPU.
- Most of the programs and data that are modifiable are storedin the RAM.
- Integrated RAM chips are available in two forms :
  (a) SRAM (Static RAM); (b) DRAM (Dynamic RAM)

SRAM	DRAM
1. In this memory, data is stored using the state of a six transistor memory cell.	1. In this RAM, each bit of a data is stored in a separate capacitor within a specific integrated circuit.
2. SRAM has less access time and faster than DRAM.	2. DRAM has higher access time and slower than SRAM.
3. It is costlier than DRAM.	3. DRAM cost is less compared to SRAM.
4. It requires constant power supply. So, it consumes more power.	4. Less power consumption because information is stored in the capacitor.
5. SRAM size is less than DRAM size.	5. DRAM size is large so more storage capacity present.

(ii)
```
# include <stdio.h>
main ( )
{
    int n, i, count=0;
    printf("Enter any number");
    scanf("%d",&n);
    if (n%2==0)
```

```
            {
                printf("even number");
            }
            else
            {
                printf("odd number");
            for (i=1; i<=n; i++)
            {
                if (n%i==0)
                {
                count ++;
                }
            }
            if (count==2)
            printf("Prime Number");
        else
        printf("Not a Prime Number");
    }
```

End of Solution

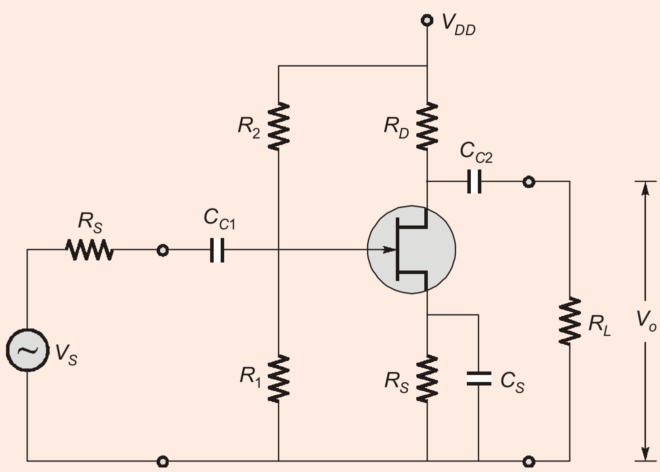
Q.7 (a) For the JFET amplifier circuit shown in the figure below, g_m = 2 mS, r_d = 200 kΩ, C_{gs} = 10 pF, C_{gd} = 2 pF, R_S = 1 kΩ, R_1 = 10 MΩ, R_2 = 100 kΩ, R_D = 5 kΩ, C_{C1} = C_{C2} = 0.1 μF. Assume output capacitor C_o = 10 pF, C_S and R_L to be very large. Find :

(i) mid-frequency gain;

(ii) lower cut-off frequency;

(iii) higher cut-off frequency;

(iv) gain-bandwidth product



[20 marks : 2023]

Solution:

Given :

$g_m = 2$ mS, $R_D = 5$ kΩ

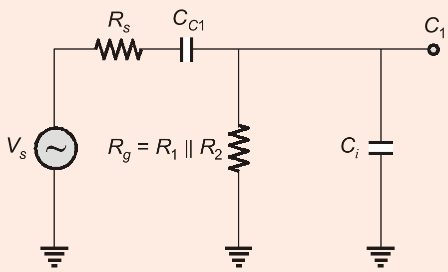
$r_d = 200$ kΩ, $C_{C1} = C_{C2} = 0.1$ μF

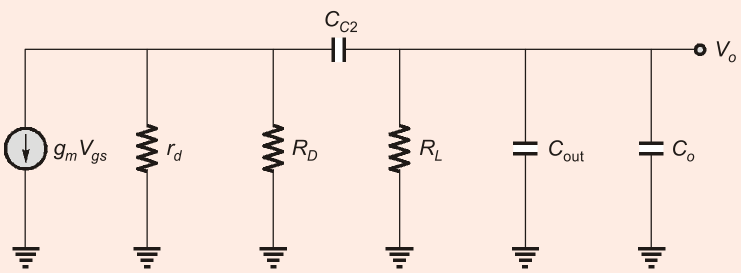
$C_{gs} = 10$ pF, $C_o = 10$ pF

$C_{gd} = 2$ pF, $R_s = 1$ kΩ

$R_1 = 10$ MΩ, $R_2 = 100$ kΩ

AC Model :





(i) Mid Frequency Gain :

$$A_v \text{ (bypass)} = -g_m R_D \parallel r_d \parallel R_L$$
$$= -2 \text{ mS } 5 \text{ K} \parallel 200 \text{ K} \parallel \infty$$
$$= -2 \text{ mS } 5 \text{ K} \parallel 200 \text{ K} \qquad (\text{Assume } R_L = \infty)$$
$$= -2 \text{ mS} \times 4.87 \text{ K}$$
$$= -9.74$$

(ii) Lower cut-off frequency

$$f_L \text{ across } C_{C1} = \frac{1}{2\pi C_{C1}(R_s + R_g)}$$
$$= \frac{1}{2 \times 3.14 \times 0.1 \mu\text{F}(1\text{ K} + 10 \text{ M}\Omega\ 100 \text{ k}\Omega)}$$
$$= 15.9 \text{ Hz}$$

$$f_L \text{ across } C_{C2} = \frac{1}{2\pi C_{C2}(R_D + R_L)}$$

where $R_D = R_D \parallel r_d;\quad R_L = \infty$

$$= \frac{1}{2\pi C_{C2}(R_D \parallel r_d + \infty)}$$

$$= \frac{1}{\infty} = 0 \text{ Hz}$$

$$f_L \text{ across } C_S = \frac{1}{2\pi C_S\left(R_s \parallel \frac{1}{g_m}\right)}$$

$$= \frac{1}{2\pi \times \infty\left(R_s \parallel \frac{1}{g_m}\right)}$$

$$= 0 \text{ Hz} \qquad (C_S \text{ is large})$$

So, lower cut off frequency,

$$f_L = 15.9 \text{ Hz}$$

(iii) Higher cut-off frequency

$$f_{H1} = \frac{1}{2\pi R_s \parallel R_g C_{in}}$$

$$= \frac{1}{2\pi R_s \parallel R_g\{C_{gs} + C_{gd}(1 + A_V)\}}$$

$$= \frac{1}{2 \times 3.14 \times 1\text{ K} \parallel 99\text{ K}\{10 \text{ pf} + (1 + 9.74)2 \text{ pf}\}}$$

$$= 5.10 \text{ MHz}$$

$$f_{H2} = \frac{1}{2\pi r_d \parallel R_D\left\{C_o + C_{gd}\left(1 + \frac{1}{A_V}\right)\right\}}$$

$$= 2.67 \text{ MHz}$$

Net f_H

$$\frac{1}{f_H} = 1.1 \times \sqrt{\frac{1}{f_{H1}^2} + \frac{1}{f_{H2}^2}}$$

$$f_H = 2.15 \text{ MHz}$$

(iv) Gain BW product

$$f_T = \frac{g_m}{2\pi(C_{gs} + C_{gd})}$$

$$= \frac{2 \text{ mS}}{2 \times 3.14(10 \text{ pF} + 2 \text{ pF})}$$

$$= 26.53 \text{ MHz}$$

End of Solution

Q.7 (b) A balanced 240 V, 3-phase voltage is applied to an unbalanced delta-connected load having the following phase impedances :

$$Z_{AB} = 25\angle 90°\ \Omega,\ Z_{BC} = 15\angle 30°\ \Omega,\ Z_{CA} = 20\angle 0°\ \Omega$$

(i) Calculate the line currents.

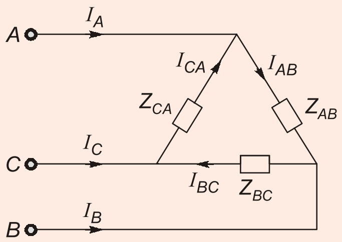
(ii) Obtain the readings of the two wattmeters whose current coils are connected in the lines *A* and *B*, and the voltage coils are connected across the line *C*.

Consider *ABC* system for supply voltage and V_{BC} as reference.

[20 marks : 2023]

Solution:

(i)



Taking V_{BC} as a reference

$$V_{BC} = 240\angle 0°$$
$$V_{CA} = 240\angle -120°$$
$$V_{AB} = 240\angle -240° \text{ or } 240\angle 120°$$

Given :

$$Z_{AB} = 25\angle 90°,\ Z_{BC} = 15\angle 30°,\ Z_{CA} = 20\angle 0°$$

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{240\angle 120°}{25\angle 90°} = 9.6\angle 30°\text{ A}$$

$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{240\angle 0°}{15\angle 30°} = 16\angle -30°\text{ A}$$

$$I_{CA} = \frac{V_{CA}}{Z_{CA}} = \frac{240\angle -120°}{20\angle 0°} = 12\angle -120°\text{ A}$$

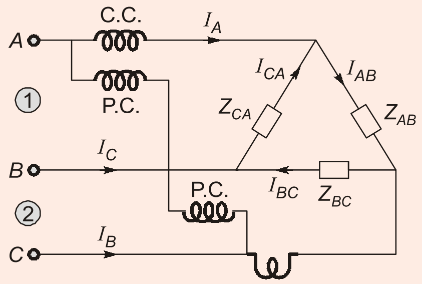
Line current will be :

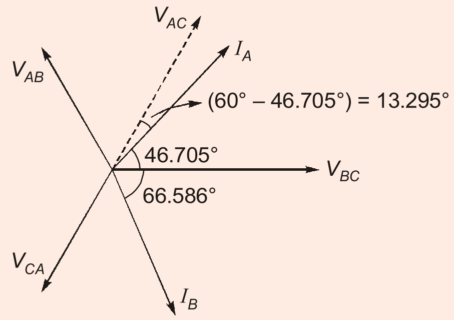
$$I_A = I_{AB} - I_{CA}$$
$$I_A = 9.6\angle 30° - 12\angle -120°$$
$$I_A = 20.87\angle 46.705\text{ A}$$

Similarly,

$$I_B = I_{BC} - I_{AB}$$
$$= 16\angle -30° - 9.6\angle 30°$$
$$I_B = 13.95\angle -66.586°$$
$$I_C = I_{CA} - I_{BC}$$
$$= 12\angle -120° - 16\angle -30°$$
$$I_C = 20\angle -173.13\text{ A}$$

(ii)





$$
\begin{aligned}
W_1 &= V_{AC} I_A \cos(13.295) \\
&= 240 \times 20.87 \times 0.973 \\
&= 4874.55 \text{ W} \\
W_2 &= V_{BC} I_B \cos(66.856) \\
&= 240 \times 13.95 \times \cos(66.586) \\
&= 1330.40 \text{ W}
\end{aligned}
$$

End of Solution

Q.7 (c) (i) Suppose that X and Y are independent random variables having the common density function

$$f(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Find the density function of the random variable $X|Y$.

(ii) A root of the equation $xe^x - 1 = 0$ lies in the interval (0.5, 1.0). Determine this root correct to three decimal places using regula-falsi method. First find how many least decimal digits are required for three decimal places accuracy.

[8 + 12 marks : 2023]

Solution:

(i) Let $Z = \dfrac{X}{Y}$

Find pdf of Z if X, Y are two R.V. and both are exponentially distributed with mean 1.

Assume $Y = y$ then, $Z = \dfrac{X}{y}$ is a scaled of X.

$$\therefore \quad f_z\left(\frac{Z}{y}\right) = |y| f_X\left(\frac{YZ}{y}\right)$$

$\therefore$ Pdf of Z is given by

$$f_Z(Z) = \int_{-\infty}^{\infty} |y| f_X\left(\frac{YZ}{y}\right) f_Y(Y)dy$$

$$= \int_{-\infty}^{\infty} |y| f_{XY}(YZ, y)dy$$

Now X, Y are independent and exponentially distributed with mean 1.

$$\therefore \quad f_Z(Z) = \int_0^{\infty} y f_X(YZ) f_Y(Y)dy \qquad (Z > 0)$$

$$= \int_0^{\infty} ye^{-yz}e^{-y}dy = \int_0^{\infty} e^{-(1+Z)y} ydy$$

Let $1 + Z = K$

$$= \int_0^{\infty} e^{-Ky} ydy$$

Let $ky = t$, $\quad dy = \frac{1}{k}dt$

$$= \int_0^{\infty} e^{-t} \frac{t}{k} \cdot \frac{1}{k} dt$$

$$= \frac{1}{k^2} \int_0^{\infty} e^{-t} t dt$$

$$= \frac{\Gamma 2}{k^2}$$

$$f_Z(Z) = \frac{1}{(Z+1)^2} \qquad (\text{for } Z > 0)$$

(ii)
$$f(x) = x^{ex} - 1 = 0$$
$$I_o = [0.5, 1.0]$$
$$f(0.5) = 0.5e^{0.5} - 1 = -0.1756$$
$$f(1) = 1e - 1 = 1.71828$$

By Regula-Falsi :

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{0.5(1.71828) - 1(-0.1756)}{1.71828 - (-0.1756)}$$

$$= 0.546369$$

$$f(x_1) = x_1 e^{x_1} - 1 = -0.056436 < 0$$

$\because$ $f(1) > 0$ and $f(x_1) < 0$

Then root of $f(x) = 0$ lies in $[x_1, 1]$

2nd approx. $x_2 = \dfrac{x_1 f(b) - bf(x_1)}{f(b) - f(x_1)} = 0.560794$

$$f(x_2) = x_2 e^{x_2} - 1 = -0.01745 < 0$$

$\because$ $f(x_2) < 0$ and $f(b) > 0$

$\therefore$ Root lies in $[x_2, b]$

3rd approx., $x_3 = \dfrac{x_2 f(b) - bf(x_2)}{f(b) - f(x_2)} = 0.565210$

$\therefore$ $f(x_3) = x_3 e^{x_3} - 1 = -0.005336 < 0$

$\because$ $f(x_3) < 0$ and $f(b) > 0$

Root lies in $[x_3, b]$

4th Approx, $x_4 = \dfrac{x_3 f(b) - bf(x_3)}{f(b) - f(x_3)} = 0.566555$

$$f(x_4) = x_4 e^{x_4} - 1 = -0.0016247 < 0$$

$\therefore$ Root lies in $[x_4, b]$

5th Approx. $x_5 = \dfrac{x_4 f(b) - bf(x_4)}{f(b) - f(x_4)} = 0.566964$

$\because$ The value of x repeating upto 3 decimals.

$\therefore$ The root of $f(x) = 0$ converges to $x = 0.567$.

To get accuracy upto 3 decimals, we need to consider 5 decimals.

Hence, the minimum number of decimals requried to get accuracy upto 3 decimals is 5.

End of Solution

Q.8 (a) (i) The power in a single-phase circuit is measured by an electrodynamometer wattmeter. The voltage across the load is 100 V and the load current is 10 A at a power factor of 0.2 lagging. The wattmeter circuit has a resistance of 3500 Ω and an inductance of 30 mH. Estimate the percentage error in the wattmeter reading when the pressure coil is connected (1) on the supply side and (2) on the load side. The current coil has a resistance of 0.1 Ω and negligible inductance. The supply frequency is 50 Hz.

(ii) The limiting errors for a four-dial resistance box are :

Units : ± 0.2%	Hundreds : ± 0.05%
Tens : ± 0.1%	Thousands : ± 0.02%

If the resistance value is set at 3525 Ω, calculate the limiting error in the resistance value.

[15 + 5 marks : 2023]

Solution:

(i) Power consumed by load,

$$P_T = VI\cos\phi$$
$$= 100 \times 10 \times 0.2 = 200 \text{ W}$$
$$\cos\phi = 0.2$$
$$\phi = \cos^{-1} 0.2 = 78.46^\circ$$
$$R_P = 3500\ \Omega,\ L = 30 \text{ mH},\ X_L = 2\pi \times 50 \times 30 \times 10^{-3}$$
$$= 9.42\ \Omega$$
$$\beta = \tan^{-1}\left(\frac{X_L}{R_P}\right) = \tan^{-1}\left(\frac{9.42}{3500}\right)$$
$$= 0.1542 \text{ rad} = 8.835^\circ$$

Consider the pressure coil connected on load side :

$$\text{Actual wattmeter reading} = (1 + \tan\phi\ .\ \tan\beta)P_T$$
$$= [1 + \tan(78.46^\circ).\tan 8.835^\circ] \times 200 = 202.63 \text{ W}$$
$$\text{Power loss in pressure coil} = \frac{V^2}{R_P} = \frac{(100)^2}{3500}$$
$$= 2.857 \text{ W}$$
$$\text{Total wattmeter reading} = 202.63 + 2.857$$
$$= 205.487 \text{ W}$$
$$\%\ \text{error} = \frac{P_W - P_T}{P_T} \times 100$$
$$= \frac{205.487 - 200}{200} \times 100$$
$$= 2.7435\%$$

Now, consider that pressure coil is connected on supply side,

$$\text{Total power} = P_T + I^2R_C$$
$$= 200 + 10^2 \times 0.1$$
$$= 210 \text{ W}$$

Impedance of load,

$$Z = \frac{V}{I} = \frac{100}{10} = 10\ \Omega$$

Resistance of load, $R_L = Z\cos\phi = 10 \times 0.2 = 2\ \Omega$

Reactance of load, $X_L = Z\sin\phi = 9.797\ \Omega$

The current coil acts as a load,

$$\text{Total resistance} = 2 + 0.1 = 2.1\ \Omega$$
$$\text{Total reactance} = 9.797\ \Omega$$

Total impedance of current coil

$$= \sqrt{(2.1)^2 + (9.797)^2}$$
$$= 10.01\ \Omega$$

$$\text{Total power factor of load} = \frac{R_T}{Z_T} = \frac{2.1}{10.01}$$
$$= 0.2097$$
$$\phi = \cos^{-1}(0.2097) = 77.89°$$
$$\tan\phi = 4.662$$

Reading of wattmeter, $P_W = (1 + \tan\phi \,.\, \tan\beta) \times P_T$
$$= (1 + 4.662 \times 0.1554) \times 200$$
$$= 344.89 \text{ W}$$

(ii) Given that Limiting errors for a four-dial resistance box :

Units : ± 0.2% Hundreds : ± 0.05%

Tens : ± 0.1% Thousands : ± 0.02%

Given resistance value : 3525 Ω

3525 Ω ⇒
$$3000 \pm 0.02\% \Rightarrow 3000 \pm 0.6$$
$$500 \pm 0.05\% \Rightarrow 500 \pm 0.25$$
$$20 \pm 0.1\% \Rightarrow 20 \pm 0.02$$
$$5 \pm 0.2\% \Rightarrow 5 \pm 0.01$$
$$3525 \pm 0.088 \Rightarrow \text{Error Value}$$

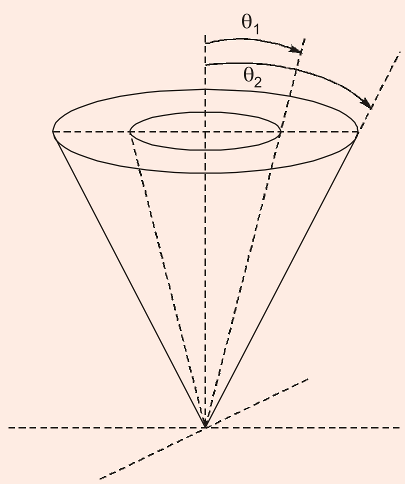
⇒ $3525 \pm 0.88 \Rightarrow$ Error Value

$$\% \text{ L.E.} = \frac{0.88}{3525} \times 100 = 0.024\%$$

⇒ $3525 \pm 0.024\%$
$$\text{L.E.} = \pm 0.024\%$$

End of Solution

Q.8 (b) In the region between the two coaxial cones with insulated vertices as shown in the figure below, the voltage at θ_1 = 30° is 0 volt and a θ_2 = 45° is 125.5 volts :



(i) Calculate the angle θ at which the voltage is 75 volts. Assume air as the dielectric in the region between the two coaxial cones.

(ii) Find the charge distribution on the conducting plane at $\theta_2 = 90°$.

[20 marks : 2023]

Solution:

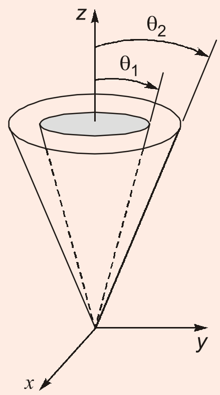
Given data : $V(\theta_1 = 30°) = 0\text{ V}$

$\Rightarrow$ $V(\theta = \theta_1) = 0$

$V(\theta_2 = 45°) = 125.5\text{ V}$

$\Rightarrow$ $V(\theta = \theta_2) = V_0$

$\Rightarrow$ $V = f(\theta)$



(i) $\because$ $\rho_V = 0$

$\therefore$ Laplace equation, $\nabla^2 V = 0$

Sp. co. sy :

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\cdot\frac{\partial v}{\partial\theta}\right)\right] = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial\theta}\left[\sin\theta\cdot\frac{\partial V}{\partial\theta}\right] = 0$$

$$\Rightarrow \quad \sin\theta\frac{dV}{d\theta} = 0$$

$$\Rightarrow \quad \frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

$$\Rightarrow \quad V = A\int\frac{d\theta}{\sin\theta} + B$$

$$= A\int\frac{d\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} + B$$

$$\Rightarrow \quad V = \frac{A}{2}\int\frac{d\theta}{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\cdot\cos^2\frac{\theta}{2}} + B$$

$$= \frac{A}{2}\int \frac{\sec^2\frac{\theta}{2}}{\tan\frac{\theta}{2}}d\theta + B$$

$\Rightarrow$ $$V = A\int \frac{\frac{1}{2}\sec^2\frac{\theta}{2}d\theta}{\tan\frac{\theta}{2}} + B$$

$$= A\int \frac{d\left(\tan\frac{\theta}{2}\right)}{\tan\frac{\theta}{2}} + B = A\left(\tan\frac{\theta}{2}\right) + B$$

At $\theta = \theta_1$: $V = 0$

$\Rightarrow$ $$0 = A\ln\left[\tan\frac{\theta_1}{2}\right] + B$$

$\Rightarrow$ $$B = -A\ln\left[\tan\frac{\theta_1}{2}\right]$$

At $\theta = \theta_2$: $V = V_o$

$\Rightarrow$ $$V_o = A\ln\left[\tan\frac{\theta_2}{2}\right] - A\ln\left[\tan\frac{\theta_1}{2}\right]$$

$\Rightarrow$ $$V_o = A\ln\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right]$$

$\Rightarrow$ $$A = \frac{V_o}{\ln\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right]}$$

and $$B = \frac{-V_o}{\ln\left[\frac{\tan\left(\frac{\theta_2}{2}\right)}{\tan\left(\frac{\theta_1}{2}\right)}\right]}\ln\left[\tan\frac{\theta_1}{2}\right]$$

Hence, $$V = \frac{V_o}{\ln\left[\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}\right]}\cdot\ln\left[\tan\frac{\theta_2}{2}\right] - \frac{V_o}{\ln\left[\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}\right]}\ln\left[\tan\frac{\theta_1}{2}\right]$$

$$\Rightarrow \qquad V = \frac{V_o}{\ln\left[\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}}\right]}\cdot \ln\left[\frac{\tan\frac{\theta}{2}}{\tan\frac{\theta_1}{2}}\right]$$

Putting the value, we get

$$V = \frac{125.5}{\ln\left[\frac{\tan 22.5°}{\tan 15°}\right]}\ln\left[\frac{\tan\frac{\theta}{2}}{\tan 15°}\right]$$

$$\Rightarrow \qquad V = 288.118\left[\ln\left(\tan\frac{\theta}{2}\right)+1.316\right]$$

Now at $\theta = \theta$, $\qquad V = 75$

$$\Rightarrow \qquad 75 = 288.118\left[\ln\left(\tan\frac{\theta}{2}\right)+1.316\right]$$

$$\Rightarrow \qquad \tan\frac{\theta}{2} = 0.348 \Rightarrow \theta = 38.36°$$

(ii) At $\theta_2 = 90°$ and $\theta_1 = 30°$

$$V = \frac{125.5}{\ln\left[\frac{\tan 45°}{\tan 15°}\right]}\ln\left[\frac{\tan\frac{\theta}{2}}{\tan 15°}\right]$$

$$= 95.295\left[\ln\left(\tan\frac{\theta}{2}\right)+1.316\right]$$

Now, $\qquad \bar{E} = -\nabla V = -\frac{1}{r}\cdot\frac{dV}{d\theta}\hat{a}_\theta = \frac{-95.295}{(r\sin\theta)}\hat{a}_\theta$

$$\Rightarrow \qquad \bar{D} = \epsilon_o \bar{E} = \frac{-95.295\,\epsilon_o}{r\sin\theta}\hat{a}_\theta$$

Now, on the plane $\theta = 90°$,

$$\therefore \qquad \rho_s = |\bar{D}| = \frac{95.295\,\epsilon_o}{r}\ \text{C/m}^2$$

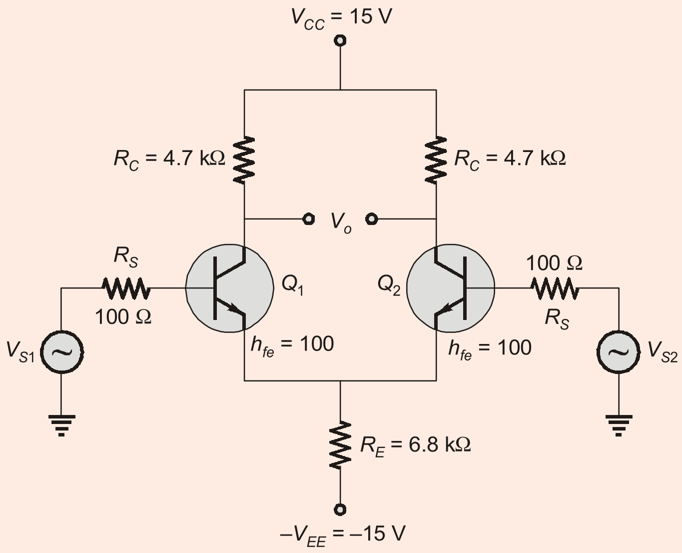
$$\Rightarrow \qquad \rho_s = \frac{8.43\times10^{-10}}{r}\ \text{C/m}^2$$

End of Solution

Q.8 (c) A dual input, balanced output differential amplifier is configured using silicon transistors which are identical having h_{ie} = 2.8 kΩ as shown in the figure below :

(i) Calculate the differential gain, common mode gain and CMRR.

(ii) What is the peak-to-peak output voltage Vo, if V_{S1} is 50 mV peak-to-peak at 2 kHz and V_{S2} is 30 mV peak-to-peak at 2 kHz?

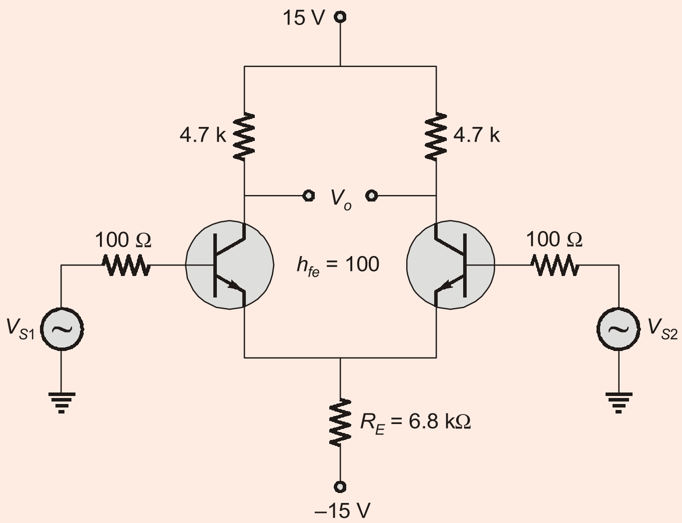


[20 marks : 2023]

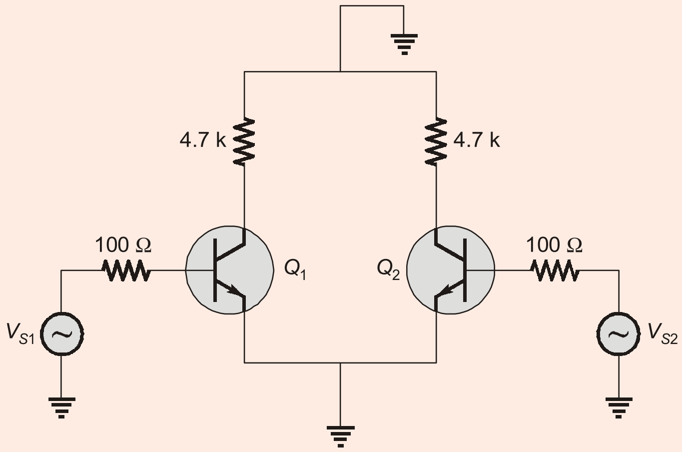
Solution:

Given dual *i*/*p* balanced o/p differential amplifier ϕ_1 and ϕ_2 are identical.

$$h_{ie} = 2.8\ \text{k}\Omega$$



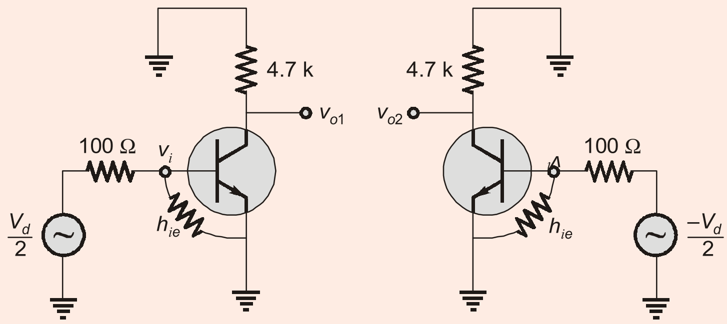
(i) Differential gain A_d :



For differential mode :

$$V_{s1} = \frac{v_d}{2}$$

$$V_{s2} = -\frac{v_d}{2}$$



Here,

$$v_d = 50 \text{ mV} - 30 \text{ mV}$$

$$= 20 \text{ mV (P-P)}$$

$$\frac{V_{o1}}{V_i} = \frac{-h_{ge}R_L}{h_{ie}} = \frac{-100 \times 4.7k}{2.8k}$$

$$\frac{V_{o1}}{V_i} = -167.85$$

$$\frac{v_i}{v_d/2} = \frac{h_{ie}}{h_{ie} + R_s} = \frac{2.8 \text{ K}}{2.8 \text{ K} + 100 \ \Omega}$$

$$= 0.965$$

$$\frac{v_i}{v_d} = \frac{0.965}{2} = 0.482$$

$$A_{d1} = \frac{V_{o1}}{v_d} = -167.85 \times 0.482 \quad \text{(Unbalanced gain)}$$

$$= -80.90$$

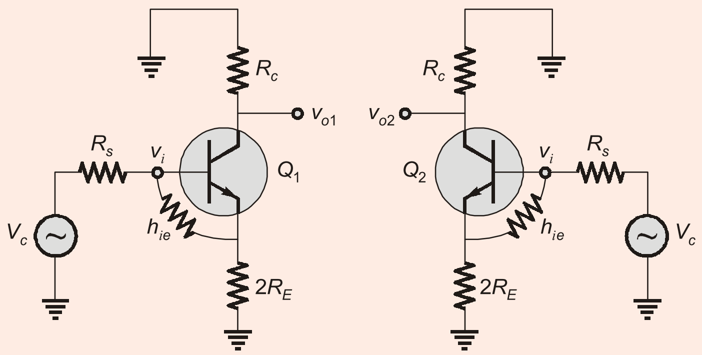
Similarly, $$A_{d2} = \frac{V_{o2}}{v_d} = 80.90 \quad \text{(Unbalanced gain)}$$

$$A_d = A_{d2} - A_{d1}$$
$$= 80.90 - (-80.90)$$
$$= 161.8 \text{ (balanced gain)}$$

$$A_d \text{ (differential gain)} = 161.8$$

Method-I :

Common mode gain AC



ϕ_1 is a unbypass circuit and ϕ_2 is similar circuit.

$$\frac{V_{o1}}{v_i} = \frac{-R_C}{r_e + 2R_E}$$

$$h_{ie} = \beta r_e$$

$$r_e = \frac{2.8k}{100} = 28\ \Omega$$

$$\frac{V_{o1}}{V_i} = \frac{-4.7k}{28\ \Omega + 2 \times 6.8k}$$

$$= -0.344$$

$$\frac{V_i}{V_c} = \frac{h_{ie} + 2(1+\beta)R_E}{R_s + h_{ie} + 2(1+\beta)R_E} = 0.99$$

$$A_c = \frac{V_{o1}}{V_c} = -0.344 \times 0.99$$

$$= -0.34$$

$$|A_c| = 0.34 \text{ (unbalanced)}$$

$$|A_c| = 0 \text{ (balanced)}$$

$$\text{CMRR} = \frac{|A_d|}{|A_c|_{\text{balanced}}}$$

$$= \infty \text{ (Theoretical)}$$

$$\text{CMRR} = \frac{A_d \text{ balanced}}{A_c \text{ Unbalanced}}$$

$$= \frac{161.8}{0.34}$$

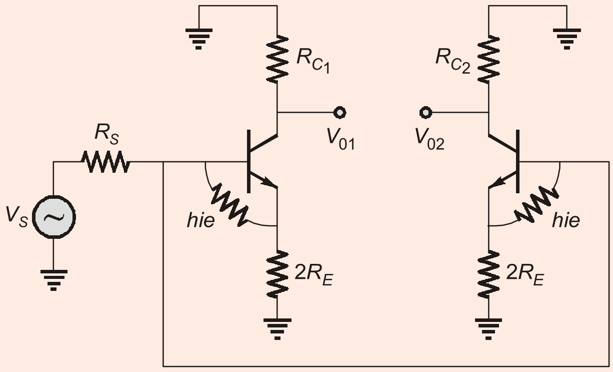
$$= 475.88 \text{ (balanced)}$$

$$\text{CMRR} = \frac{A_d \text{ unbalanced}}{A_c \text{ unbalanced}}$$

$$= \frac{80.90}{0.34} = 237.94 \text{ (unbalanced)}$$

Method-II :

To calculate common mode gain



$$\frac{V_{o1}}{V_s} = \frac{-h_{fe}I_bR_c}{2I_bR_s + h_{ie}I_b + (1+h_{fe})I_b \times 2R_E}$$

$$\frac{V_{o1}}{V_s} = \frac{-h_{fe}R_{c1}}{2R_s + h_{ie} + 2(1+h_{fe})R_E}$$

$$\frac{V_{o2}}{V_s} = \frac{-h_{fe}R_{c2}}{2R_s + h_{ie} + 2(1+h_{fe})R_E}$$

For balanced output

$$A_c = \frac{V_{o1} - V_{o2}}{V_s}$$

$$A_c = \frac{h_{fe}(R_{c2} - R_{c1})}{2R_s + h_{ie} + 2(1+h_{fc})R_E}$$

If $R_{c1} = R_{c2}$, then $A_c = 0$

If $R_{c1} \neq R_{c2}$, then $A_c \neq 0$

Assume $R_{c1} = R_c + 1\% = 1.01R_c$

$R_{c2} = R_c - 1\% = 0.99R_c$

$$A_c = \frac{h_{fe}(0.99R_c - 1.01R_c)}{2R_s + h_{ie} + 2(1+h_{fe})R_E} = -6.828 \times 10^{-3}$$

$$\text{CMRR} = \left|\frac{A_D}{A_C}\right| = \frac{161.8}{6.828 \times 10^{-3}} = 23.69 \times 10^3$$

End of Solution