

Electronics Engineering

Communication Systems

Comprehensive Theory

with Solved Examples and Practice Questions



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Communication Systems

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Communication Systems

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Introduction to Communication Systems

1.1 Historical Sketch

The development of communication technology has proceeded in step with the development of electronic technology as a whole. For example, the demonstration of telegraphy by Joseph Henry in 1832 and by Samuel F.B. Morse in 1838 followed hard on the discovery of electromagnetism by Oersted and Ampere early in 1820's. Similarly, Hertz's verification late in the 1880's of Maxwell's postulation (1873) predicting the wireless propagation of electromagnetic energy led within 10 years of the radio-telegraph experiments of Marconi and Popov. The invention of diode by Fleming in 1904 and of triode by deForest in 1906 made possible the rapid development of long distance telephony, both by radio and wireless.

1.2 Why Study Communication

The rapidly changing face of technology necessitates learning of new technology. Today the question is no longer in the field of invention but of innovation. The question today in the twenty first century is not how to transmit data from point A to point B but how efficiently can we do it. To be able to answer this question, first we should be able to diagnose the problem. This can be done only by studying communication from the beginning to its modern form.

1.3 What is Communication

In the most fundamental sense, communication involves implicitly the transmission of information from one point to another through a succession of processes, as described here:

1. The generation of a message signal: voice, music, picture, or computer data.
2. The description of that message signal with a certain measure of precision, by a set of symbols: electrical, aural, or visual.
3. The encoding of these symbols in a form that is suitable for transmission over a physical medium of interest.
4. The transmission of the encoded symbols to the desired destination.
5. The decoding of the reproduction of the original symbols.
6. The re-creation of the original message signal, with a definable degradation in quality; the degradation is caused by imperfections in the system.

There are, of course, many other forms of communication that do not directly involve the human mind in real time. For example, in computer communications involving communication between two or more computers, human decisions may enter only in setting up the programs or commands for the computer, or in monitoring the results.

1.4 Communication Model

The study of communication becomes easier, if we break the whole subject of communication in parts and then study it part by part. The whole idea of presenting the model of communication is to analysis the key concepts used in communication in isolated parts and then combining them to form the complete picture.

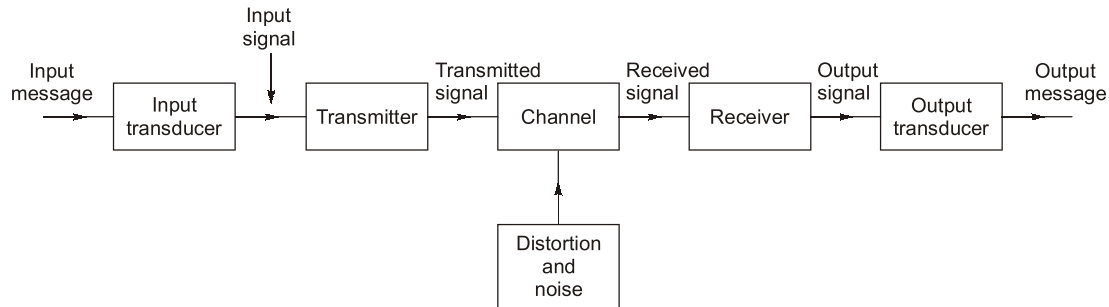


Figure-1.1: Model of communication system

Source

The source originates a message, such as a human voice, a television picture, an e-mail message, or data. If the data is non-electric (e.g., human voice, e-mail text, television video), it must be converted by an **input transducer** into an electric waveform referred to as the **baseband signal** or **message signal** through physical devices such as a microphone, a computer keyboard or a CCD camera.

Transmitter

The transmitter modifies the baseband signal for efficient transmission. The transmitter may consist of one or more subsystems: an A/D converter, an encoder and a modulator. Similarly, the receiver may consists of a demodulator, a decoder and a D/A converter.

Channel

The channel is a medium of choice that can convey the electric signals at the transmitter output over a distance. A typical channel can be a pair of twisted copper wires (telephone and DSL), coaxial cable (television and internet), an optical fibre or a radio link. Channel may be of two types.

1. **Physical channel:** When there is a physical connection between the transmitter and receiver through wires. eg. coaxial cable.
2. **Wireless channel:** When no physical channel is present and transmission is through air. eg. mobile communication.

It is inevitable that the signal will deteriorate during the process of transmission and reception as a result of some distortion in the system, or because of the introduction of noise, which is unwanted energy, usually of random character, present in a transmission system, due to a variety of causes. Since noise will be received together with the signal, it places a limitation on the transmission system as a whole. When noise is severe, it may mask a given signal so much that the signal becomes undetectable and therefore useless. Noise may interfere with signal at any point in a communications system, but it will have its greatest effect when the signal is weakest. This means that noise in the channel or at the input to the receiver is the most noticeable.

Receiver

The receiver reprocesses the signal received from the channel by reversing the signal modifications made at the transmitter and removing the distortions made by the channel. The receiver output is fed to the output transducer, which converts the electric signal to its original form i.e. the message signal.

Destination

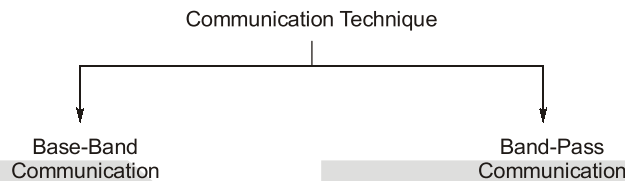
The destination is the unit to which the message is communicated.

1.5 Modes of Communication

There are two basic modes of communication:

1. **Broadcasting**, which involves the use of a single powerful transmitter and numerous receivers that are relatively inexpensive to build. Here information-bearing signals flow only in one direction.
2. **Point-to-point communication**, in which the communication process takes place over a link between a single transmitter and a receiver. In this case, there is usually a bidirectional flow of information-bearing signals, which requires the use of a transmitter and receiver at each end of the link.

1.5.1 Communication Technique



1. Base Band Communication:

It is generally used for short distance communication. In this type of communication message is directly sent to the receiver without altering its frequency.

2. Band Pass Communication:

It is used for long distance communication. In this type of communication, the message signal is mixed with another signal called as the carrier signal for the process of transmission. This process of adding a carrier to a signal is called as modulation.

1.5.2 Need of Modulation

1. To avoid the mixing of signals

All messages lies within the range of 20 Hz - 20 kHz for speech and music, few MHz for video, so that all signals from the different sources would be inseparable and mixed up. In order to avoid mixing of various signals, it is necessary to translate them all to different portions of the electromagnetic spectrum.

2. To decrease the length of transmitting and receiving antenna

For a message at 10 kHz, the antenna length 'l' for practical purposes is equal to $\lambda/4$ (from antenna theory) i.e.,

$$\lambda = \frac{3 \times 10^8}{10 \times 10^3} = 3 \times 10^4 \text{ m}$$

and
$$l = \frac{\lambda}{4} = \frac{3 \times 10^4}{4} = 7500 \text{ m}$$

An antenna of this size is impractical and for a message signal at 1 MHz

$$\lambda = \frac{3 \times 10^8}{10^6} = 300 \text{ m}$$

and
$$l = \frac{\lambda}{4} = 75 \text{ m (practicable)}$$

3. To allow the multiplexing of signals

By translating all signals from different sources to different carrier frequency, we can multiplex the signals and able to send all signals through a single channel.

4. To remove the interference
5. To improve the quality of reception i.e. increasing the value of S/N ratio
6. To increase the range of communication

1.6 Types of Modulation

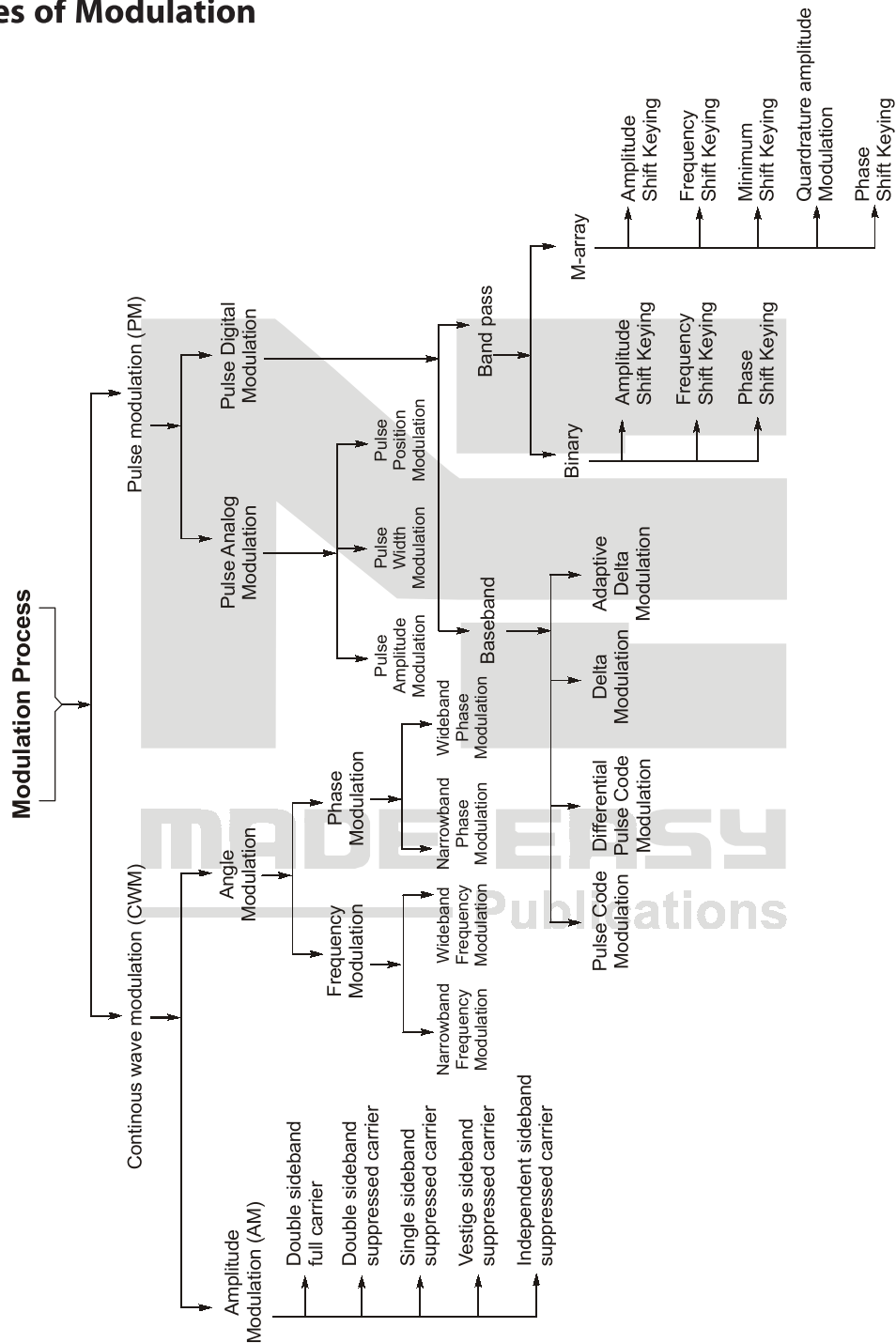


Figure-1.2

Basics of Signal and System

Introduction

Just as a carpenter requires proper set of tools before he can sit down to make a piece of furniture, in a similar manner a communication engineer needs to know about signals before he can start the process of learning communication.

2.1 Signal and System

The communication technology can be conveniently broken down into three interacting parts.

- Signal processing operations performed.
- The device that performs these operations.
- The underlying physics.

Thus to study the basic form of modulation and signal processing used in the communication it will be fruitful to have a quick review of the concepts of signal and system.

2.1.1 Some Basic Signals

It will be very helpful to study some signals before hand, so that the analysis of the communication, system becomes easier. Some important and frequently used signals and their properties are mentioned in this section.

The Impulse Signal

1. Impulse signal (Dirac delta function):

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

and
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

2. Unit impulse signal:

$$\delta[n] = \begin{cases} 1; & n = 0 \\ 0; & n \neq 0 \end{cases}$$

Properties of Impulse Function

Product property

$$x(t) \delta(t) = x(0) \delta(t)$$

Similarly, $x(t) \delta(t - \alpha) = x(\alpha) \delta(t - \alpha)$

Shifting property

$$\int_{-\infty}^{\infty} x(t) \delta(t) dt = x(0) \quad -\infty \leq t \leq \infty$$

Similarly,

$$\int_{-\infty}^{\infty} x(t) \delta(t - \alpha) dt = x(\alpha) \quad -\infty \leq \alpha \leq \infty$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

Scaling property

$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$

Example 2.1

signal.

Find the impulse function form if $x(t) = 4t^2 \delta(2t - 4)$, where $x(t)$ is an arbitrary

Solution:

$$\begin{aligned} x(t) &= 4t^2 \delta(2t - 4) \\ &= 4t^2 \delta\{2(t - 2)\} \\ &= 4t^2 \cdot \frac{1}{2} \delta(t - 2) \quad \dots \text{from scaling property} \\ &= 2t^2 \delta(t - 2) \end{aligned}$$

Now, from product property we have,

$$x(t) \delta(t - \alpha) = x(\alpha) \delta(t - \alpha)$$

so,

$$\begin{aligned} x(t) &= 2t^2 \Big|_{t=2} \cdot \delta(t - 2) \\ &= 8 \delta(t - 2) \end{aligned}$$

Example 2.2

Let $\delta(t)$ denote the delta function. The value of the integral

$$\int_{-\infty}^{\infty} \delta(t) \cos\left(\frac{3t}{2}\right) dt \text{ is}$$

- (a) 1 (b) -1 (c) 0 (d) $p/2$

Solution: (a)

We know,

$$\int_{-\infty}^{\infty} \delta(t) x(t) dt = x(0)$$

So here,

$$\int_{-\infty}^{\infty} \delta(t) \cos\left(\frac{3t}{2}\right) dt = \cos 0 = 1$$

Do you know? Impulse signals do not occur naturally but they are important functions providing a mathematical frame work for the representation of various processes and signals. These come under a special class of functions known as generalized functions.

Gate Function/Rectangular Pulse

Let us consider a rectangular pulse as shown in figure below:

$$x(t) = \begin{cases} A \operatorname{rect}(t) = A, & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

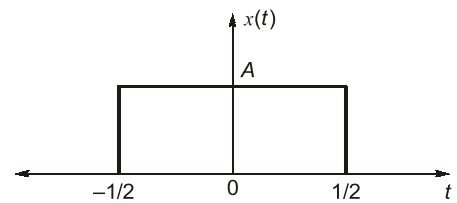


Figure-2.1

$$x(t) = \begin{cases} A \operatorname{rect}\left(\frac{t}{\tau}\right) = A, & \text{for } -\frac{\tau}{2} < t < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$$

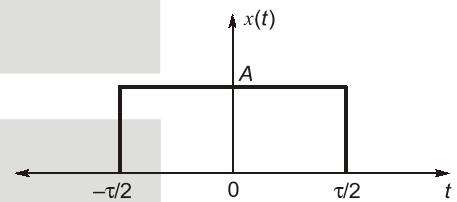


Figure-2.2

Step Signal

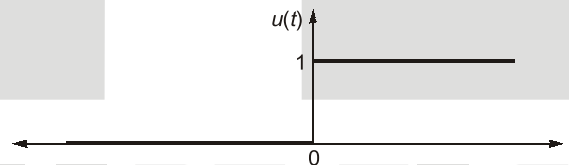


Figure-2.3: Continuous-time version of the unit-step function of unit amplitude

The continuous-time version of the unit-step function is defined by

$$u(t) = \begin{cases} 1; & t > 0 \\ 0; & t < 0 \end{cases}$$

NOTE



- Figure depicts the unit-step function $u(t)$. It is said to exhibit discontinuity at $t = 0$, since the value of $u(t)$ changes instantaneously from 0 to 1 when $t = 0$. It is for this reason that we have left out the equal sign in equation; that is $u(0)$ is undefined.
- Unit step function denote sudden change in real time and a frequency or phase selectivity in frequency domain.

There is one more definition of unit step function.

$$u(t) = \begin{cases} 0 & ;t < 0 \\ 1/2 & ;t = 0 \\ 1 & ;t > 0 \end{cases}$$

Do you know? The unit-step function $u(t)$ may also be used to construct other discontinuous waveforms. The value at $t = 0$ gives rise to Gibb's phenomenon when unit step function is constructed by sinusoidal signals.

Sampling/Interpolating/Sinc Function

The function $\frac{\sin x}{x}$ is the "sine over argument" function and it is denoted by "sinc (x)". It is also known as "filtering function".

Mathematically,

$$\begin{aligned} \text{sinc}(x) &= \frac{\sin \pi x}{\pi x} \\ &= \text{Sa}(\pi x) \end{aligned}$$

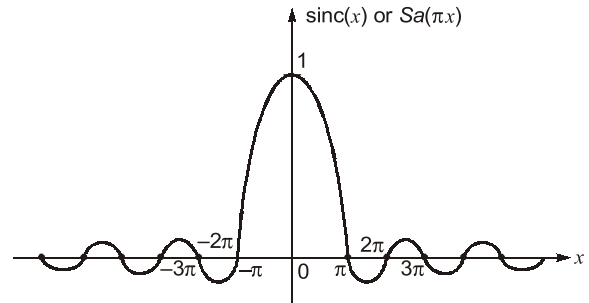


Figure-2.4: Sinc Function

Do you know? Just like impulse function $\delta(x)$ is also a conceptual function since it can not be realized.

2.1.2 Energy Signals and Power Signals

$x(t)$ is an energy signal

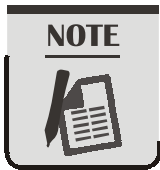
For an energy signal, energy is finite while power is zero.

$$0 < E < \infty \quad \text{and} \quad P = 0$$

where 'E' is the energy and 'P' is the power of the signal $x(t)$.

For a continuous-time signal (CTS),

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$



If $x(t) \longrightarrow E$, [where, E is energy of $x(t)$]

then $x\left(\frac{t}{\alpha}\right) \longrightarrow \alpha E$

$x(\alpha t) \longrightarrow \frac{E}{\alpha}$

$ax(t) \longrightarrow a^2 E$

$x(t)$ is a "Power Signal"

if, $0 < P < \infty$ and $E = \infty$

where $E =$ Energy of signal $x(t)$

$P =$ Power of signal $x(t)$

Almost all the practical periodic signals are "power signals", since their average power is finite and non-zero.

For a CTS, the average power of a signal $x(t)$ is,

$$P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt$$

NOTE

- If $x(t) = A \cos \omega t$ or $A \sin \omega t$

$$P_x = \frac{A^2}{2}$$

- If $x(t) = Ae^{\pm j\omega t}$
 $\Rightarrow P_x = A^2$
- If $x(t) = A$
 $\Rightarrow P_x = A^2$

- If $x(t) \longrightarrow P$

$$\text{then } x\left(\frac{t}{\alpha}\right) \longrightarrow P$$

$$x(\alpha t) \longrightarrow P$$

$$ax(\alpha t) \longrightarrow a^2P$$

- For an **unit step signal**, $x(t) = u(t)$

$$P_x = \frac{1}{2}$$

2.2 Time Domain and Frequency Domain Representation of a Signal

A signal $x(t)$ can be represented in terms of relative amplitude of various frequency components present in signal. This is possible by using exponential Fourier series. This is a frequency domain representation of the signal. The time domain representation specifies a signal value at each instant of time. This means that a signal $x(t)$ can be specified in two equivalent ways:

- (i) Time domain representation; where $x(t)$ is represented as a function of time. Graphical time domain representation is termed as *waveform*.
- (ii) The frequency domain representation; where the signal is represented graphically in terms of its spectrum.

Any of the above two representations uniquely specifies the function, i.e. if the signal is specified in time domain, we can determine its spectrum. Conversely, if the spectrum is specified, we can determine the corresponding time domain signal. In order to determine the function in frequency domain, it is necessary that both amplitude spectrum and phase spectrum are specified.

Remember

- In many cases, the spectrum is either real or imaginary, as such, only an amplitude plot is enough as all frequency components have identical phase relation.
- We use both the conventions depending upon the problem we are studying.
- If we want to analyze the signal at our perspective, it is convenient to see signal in its time domain form, but if we want to process the signal through an LTI system, the frequency domain approach becomes much fruitful.

2.2.1 Decomposition of Signals

It will be fruitful for us if we can devise some technique that will allow us to break any unknown signal into some standard and known signal set. There are two methods of doing this

- (1) Any signal can be broken down into an infinite set of impulse signals. This process leads to the time domain approach of signal and system.
- (2) It can be broken down into an infinite set of orthogonal signals. This leads to frequency domain description of signal and system.

Misconception: Not any representation of signal as set of impulse or exponential function is considered as signal decomposition.

Consider a periodic signal $x(t) = \begin{cases} 1; & 0 \leq t \leq 1 \\ -2; & 1 < t < 2 \end{cases}$ with time period $T = 2$.

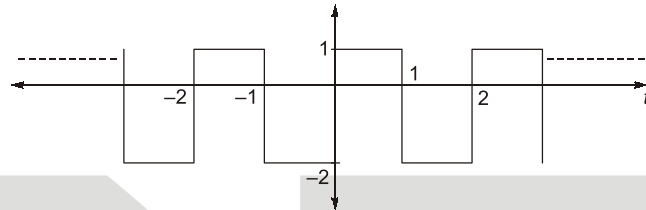


Figure-2.5

The derivative of this signal is related to the impulse train $g(t) = \sum_{k=-\infty}^{\infty} [a_k \delta(t - k) + b_k \delta(t - 2k)]$ with period $T = 2$. [where, $a_k = 3$, $b_k = -3$]

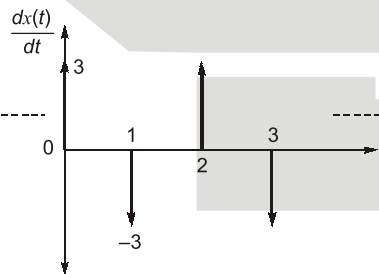


Figure-2.6

NOTE: The problem here is we have to define a differentiator in order to represent the time domain signal as set of impulses. Thus we need an alternative way of representing the signal.

2.3 Signals Versus Vectors

There is a strong connection between signals and vectors. Signals that are defined for only a finite number of time instants (say N) can be written as vectors (of dimension N). Thus, consider a signal $g(t)$ defined over a closed time interval $[a, b]$. Let us pick N points uniformly on the time interval $[a, b]$ such that

$$t_1 = a, t_2 = a + \epsilon, t_3 = a + 2\epsilon, t_N = a + (N - 1)\epsilon = b, \epsilon = \frac{b - a}{N - 1}$$

Then we can write a signal vector g as an N -dimensional vector

$$g = [g(t_1) \ g(t_2) \ \dots \ g(t_N)]$$

This relationship clearly shows that continuous time signals are straight forward generalizations of finite dimension vectors. Thus, basic definitions and operations in a vector space can be applied to continuous time signals as well. In a vector space, we can define the inner (dot or scalar) product of two real-valued vectors x and g as

$$\langle x, g \rangle = \|g\| \cdot \|x\| \cos \theta$$

When θ is the angle between vectors x and g .

By using this definition, we can express $\|x\|$, the length (norm) of a vector x as

$$\|x\|^2 = \langle x, x \rangle$$

Remember: This concept forms the basis of digital communication system.

2.3.1 Decomposition of a Signal and Signal Components

The concepts of vector component and orthogonality can be directly extended to continuous time signals. Consider the problem of approximating a real signal $g(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$.

$$e = \frac{\int_{t_1}^{t_2} g(t)x(t)dt}{\int_{t_1}^{t_2} x(t)dt} = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x(t)dt \quad (\text{where, } E_x = \text{Energy of signals})$$

NOTE



If a signal $g(t)$ is approximated by another signal $x(t)$ as

$$g(t) = ex(t)$$

then the optimum value of e that minimizes the energy of the error signal in this approximation is given by above equation.

2.4 Orthogonal Signal Set

In this section we show a way of representing a signal as a sum of orthogonal set of signals. Infact, the signals in this orthogonal set form a basis for the specific signal space. Here again we can benefit from the insight gained from a similar problem in vectors. We know that a vector can be represented as a sum of orthogonal vectors, which form the coordinate system of a vector space.

2.4.1 Orthogonal Signal Space

We continue with signal approximation problem, using clues and insights developed for vector approximation. We define orthogonality of a signal set $x_1(t), x_2(t), \dots, x_n(t)$ over a time domain Θ .

$$\int x_m(t)x_n^*(t)dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

If all signal energies are equal to unity $E_n = 1$, then the set is normalized and is called an orthonormal set.

An orthogonal set can always be normalized by dividing $x_n(t)$ by $\sqrt{E_n}$ for all n . A signal $g(t)$ over the time domain Θ can be represented by a set of N mutually orthogonal signals $x_1(t), x_2(t), \dots, x_N(t)$:

$$\begin{aligned} g(t) &\simeq C_1 x_1(t) + C_2 x_2(t) + \dots + C_N x_N(t) \\ &= \sum_{n=1}^N C_n x_n(t) \end{aligned}$$

It can be shown that E_n , the energy of the error signal $e(t)$ in this approximation, is minimized if we choose

$$C_n = \frac{\int_{t \in \Theta} g(t)x_n^*(t)dt}{\int_{t \in \Theta} |x_n(t)|^2 dt} = \frac{1}{E_n} \int g(t)x_n^*(t)dt \quad n = 1, 2, \dots, N$$

NOTE : If the orthogonal set is complete, then the error energy $E_n \rightarrow 0$.

Remember: Orthogonal signal set forms the basis signals set for the representation of signals just like the coordinate axis are used in maps to represent a point.

2.5 The Fourier Series

Let $g_p(t)$ represent a periodic signal with period t_0 . With the help of Fourier series, we are able to resolve the signal into infinite sum of sine and cosine signals. An alternative way of saying this definition can be we break up the periodic signal $g_p(t)$ into an infinite sum of harmonics which helps us to define the signal in the frequency domain.

There are two ways to represent the Fourier series.

- (1) Trigonometric Fourier series
- (2) Exponential Fourier series

2.5.1 Trigonometric Form of Fourier Series

The expression may be expressed as.

$$g_p(t) = a_0 + 2 \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nt}{T_0}\right) + b_n \sin\left(\frac{2\pi nt}{T_0}\right) \right]$$

The above equation is termed as **Synthesis Equation**. Thus the periodic signal $g_p(t)$ is now represented as set of orthogonal signal.

Where the coefficients a_n and b_n represents the unknown amplitudes of the cosine and sine forms, respectively. The quantity with n/T_0 represents the n^{th} harmonic of the fundamental frequency $f_0 = \frac{1}{T_0}$.

Now, we need to calculate the coefficient a_n , b_n and a_0 . The equation which are used to calculate these value are known as **Analysis Equation**. The equations for the trigonometric Fourier series are given below.

$$a_0 = \frac{1}{T_0} \int_0^{T_0} g_p(t) dt$$

$$a_n = \frac{1}{T_0} \int_0^{T_0} g_p(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt$$

$$b_n = \frac{1}{T_0} \int_0^{T_0} g_p(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt$$

2.5.2 Exponential Form of Fourier Series

In the above expression of trigonometric Fourier series the orthogonal set used for the creation of the periodic signal were sine and cosine terms. In the similar way we can represent the periodic signal in terms of an infinite set of orthogonal complex exponential signals.

Synthesis equation

$$g_p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \quad n = 0, 1, 2, 3, \dots$$

Analysis equation

$$C_n = \frac{1}{T_0} \int_0^T g_p(t) e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} g_p(t) e^{-\frac{j2\pi n t}{T_0}} dt$$

- NOTE :**
- C_n and C_{-n} are in complex-conjugate pair when the signal $g_p(t)$ is a real signal i.e. $C_n = C_{-n}^*$
 - C_n s are the spectral amplitudes of the spectral component $C_n e^{j2\pi n f_0 t}$.

2.5.3 Frequency Spectrum of Non-sinusoidal Wave

Amplitude of wave is 'A' and repetition rate is $\omega/2\pi$ per second, then.

(a) Square wave: $g_p(t) = \frac{4A}{\pi} \left(\cos \omega t - \frac{1}{3} \cos 3\omega t + \frac{1}{5} \cos 5\omega t - \frac{1}{7} \cos 7\omega t + \dots \right)$

(b) Triangular wave: $g_p(t) = \frac{4A}{\pi^2} \left(\cos \omega t - \frac{1}{9} \cos 3\omega t + \frac{1}{25} \cos 5\omega t + \dots \right)$

(c) Sawtooth wave: $g_p(t) = \frac{2A}{\pi} \left(\sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t - \frac{1}{4} \sin 4\omega t + \dots \right)$

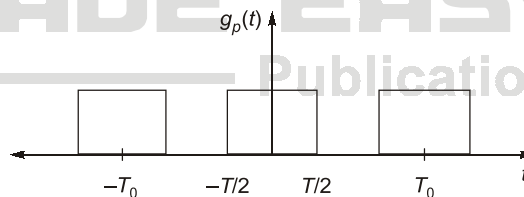
NOTE

From the analysis equation in trigonometric Fourier series we conclude that:

- The trigonometric Fourier series of an even function of time contains only D.C. term and cosine terms.
- The trigonometric Fourier series of an odd function of time contains only sine terms.

Example 2.3

Given a periodic signal $g_p(t)$ as shown the figure below.



Find the complex Fourier coefficient C_n .

Solution:

$$g_p(t) = \begin{cases} A & -T/2 \leq t \leq T/2 \\ 0 & \text{for the remainder of the period} \end{cases}$$

Now,

$$C_n = \frac{1}{T_0} \int_{-T/2}^{T/2} A \exp\left(\frac{-i2\pi n t}{T_0}\right) dt$$

\therefore

$$C_n = \frac{A}{n\pi} \sin\left(\frac{n\pi T}{T_0}\right); \quad n = 0, \pm 1, \pm 2 \dots$$