POSTAL Book Package 2023

ESE

Electronics Engineering

Conventional Practice Sets

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Contents

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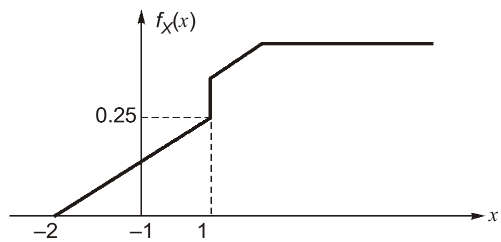
CHAPTER

Theory of Random Variable and Noise

Q.1 Define PDF and summarise its important properties. Also calculate the probability of outcome of a Random Variable (RV) X having $X \le 1$ for the following PDF curve of RV as shown.



Solution:

Probability density function specifies the probability of a random variable taking a particular value.

The Probability Density Function (PDF) which is generally denoted by $f_X(x)$ or $P_X(x)$ or $\rho_X(x)$ is defined in terms of the Cumulative Distribution Function (CDF) $F_X(x)$ as,

$$\boxed{\text{PDF} = f_X(x) = \frac{d}{dx}F_X(x)} \quad \text{...(i)}$$

The PDF has the following properties:

(i) $f_X(x) \ge 0$ for all x

This results from the fact that probability cannot be negative. Also, $F_X(x)$ increases monotonically, as x increases, more outcomes are included in the prob. of occurrence represented by $F_X(x)$.

(ii) Area under the PDF curve is always equal to unity.

i.e. $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

(iii) The CDF is obtained by the result

$$\text{CDF} = \int_{-\infty}^{x} f_X(x)dx$$

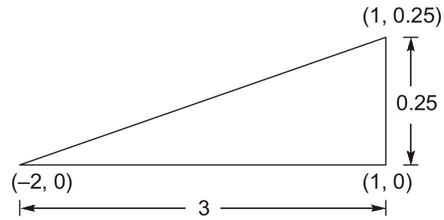
(iv) Probability of occurrence of the value of random variable between the limits of x_1 and x_2 is given by,

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} f_X(x)dx$$

Now consider the given PDF curve, since we have to find $P(x \le 1)$ so,

Equation for the PDF curve for $x \le 1$ is,

$$f_X(x) = \left(\frac{1}{12}x + \frac{1}{6}\right)$$



Now, $P(x \leq 1)$

$$= P(-2 < x < 1) = \int_{-2}^{1}\left(\frac{1}{12}x + \frac{1}{6}\right)dx = \left[\frac{1}{12}\cdot\frac{x^2}{2} + \frac{1}{6}x\right]_{-2}^{1} = \frac{3}{8}$$

$$\therefore \qquad P(x \leq 1) = \frac{3}{8}$$

Q.2 Find the cumulative distribution function $F(x)$ corresponding to the PDF $f(x) = \dfrac{1}{\pi(1+x^2)}, -\infty < x < \infty.$

Solution:

Given
$$f(x) = \frac{1}{\pi(1+x^2)}, -\infty < x < \infty$$
$$F(x) = P(X \leq x)$$
$$= \int_{-\infty}^{x} f(x)dx = \frac{1}{\pi}\int_{-\infty}^{x}\frac{dx}{1+x^2} = \frac{1}{\pi}\left[\tan^{-1}x\right]_{-\infty}^{x} = \frac{1}{\pi}\left(\frac{\pi}{2} + \tan^{-1}x\right)$$

Q.3 Given the random variable X with density function

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the PDF of $Y = 8X^3$.

Solution:

$y = 8x^3$ is an increasing function in (0, 1)

Given $\quad y = 8x^3$

$$\Rightarrow \quad x^3 = \frac{y}{8}$$

$$\Rightarrow \quad x = \left(\frac{y}{8}\right)^{1/3} = \frac{1}{2}y^{1/3}$$

and $\quad f_X(x) = 2x, \qquad 0 < x < 1$

$$f_X(y) = \frac{2y^{1/3}}{2} = \frac{y^{1/3}}{3}$$

$$f_Y(y) = \qquad \ldots(i)$$

Given $\quad x = \left(\frac{y}{8}\right)^{1/3} = \frac{1}{2}y^{1/3} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{6}y^{-2/3}$

Using it in (i)

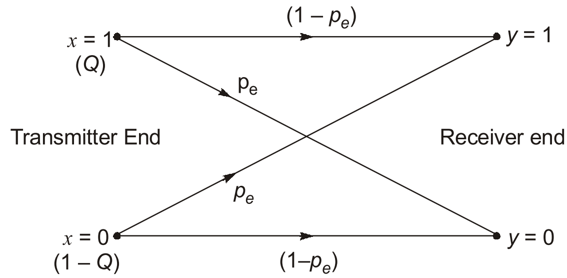
$$f_Y(y) = y^{1/3}\,\frac{1}{6}y^{-2/3} = \frac{1}{6}y^{-1/3} = \frac{1}{6}\,\frac{1}{y^{1/3}} = \frac{1}{6\sqrt[3]{y}}$$

The range for x is $0 < x < 1$

When $x = 0,\ y = 8 \times 0 = 0$ and $x = 1,\ y = 8 \times 1^3 = 8$

$$f_Y(y) = \frac{1}{6\sqrt[3]{y}}, \qquad 0 < y < 8$$

Q.4 A BSC (Binary Symmetric Channel) error probability is P_e. The probability of transmitting '1' is Q, and that of transmitting '0' is $(1 - Q)$ as in figure below. Calculate the probabilities of receiving 1 and 0 at the receiver?



Solution:

If x and y are the transmitted digit and the received digit respectively, then for a BSC,

$$P_{y|x}(0|1) = P_{y|x}(1|0) = P_e$$

$$P_{y|x}(0|0) = P_{y|x}(1|1) = 1 - P_e$$

Also, $P_x(1) = Q$ and $P_x(0) = 1 - Q$

We have to find, $P_y(1)$ and $P_y(0) = ?$

$\therefore$ $P_y(1) = P_x(0)\,P_{y|x}(1|0) + P_x(1)\,P_{y|x}(1|1) = (1 - Q)P_e + Q(1 - P_e)$

also, $P_y(0) = P_x(0)P_{y|x}(0|0) + P_x(1)\,P_{y|x}(0|1) = (1 - Q)(1 - P_e) + QP_e$

Q.5 For the triangular distribution

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Find the mean and variance.

Solution:

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x \cdot x\,dx + \int_1^2 x(2-x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{3} + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^1 x^2 x\,dx + \int_1^2 x^2(2-x)\,dx$$

$$= \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)dx = \left[\frac{x^4}{4}\right]_0^1 + \left[2\left(\frac{x^3}{3}\right) - \frac{x^4}{4}\right]_1^2$$

$$= \frac{1}{4} + \left[\left(\frac{16}{3} - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right)\right] = \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{7}{6} - (1)^2 = \frac{1}{6}$$

Q.6 The joint density function of two continuous random variables is given by

$$f(x, y) = \begin{cases} xy/8, & 0 < x < 2,\ 1 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

Find (a) $E(X)$, (b) $E(Y)$ and (c) $E(2X + 2Y)$.

Solution:

(a) $$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x, y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} x(xy/8)\,dx\,dy = \frac{4}{3}$$

(b) $$E(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x, y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} y(xy/8)\,dx\,dy = \frac{13}{6}$$

(c) $$E(2X+3Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2x+3y)\,dx\,dy = \int_{x=0}^{2}\int_{y=1}^{3} (2x+3y)(xy/8)\,dx\,dy = \frac{55}{6}$$

Q.7 Let z be a random variable with probability density function $f_Z(z) = \frac{1}{2}$ in the range $-1 \le z \le 1$. Let the random variable $x = z$ and the random variable $y = z^2$. Obviously x and y are not independent since $x^2 = y$. Show that x and y are uncorrelated.

Solution:

We have, $$E(z) = \int_{-1}^{1} z \cdot f_Z(z)\,dz$$

$\Rightarrow$ $$E(z) = \frac{1}{4}\left[z^2\right]_{-1}^{1} = 0$$

Since, $x = z$, so $\quad E(x) = E(z) = 0$

Since, $y = z^2$ so $\quad E(y) = E(z^2)$

So that, $$E(y) = \int_{-1}^{1} \frac{1}{2} z^2\,dz = \frac{1}{6}\left[z^3\right]_{-1}^{1} = \frac{1}{3}$$

We know that, the co-variance 'μ' of two RVs x and y is defined as,

$$\mu = E\{(x - m_x)(y - m_y)\}$$

$$= E\left\{(x)\left(y - \frac{1}{3}\right)\right\} = E\left\{xy - \frac{1}{3}x\right\} = E\left\{z^3 - \frac{z}{3}\right\} = \int_{-1}^{1}\frac{1}{2}\left(z^3 - \frac{z}{3}\right)dz$$

$$\mu = 0$$

Now, correlation coefficient between the variables x and y is defined by quantity 'ρ' as,

$$\rho = \frac{\mu}{\sigma_x \sigma_y} = 0$$

So, we can say that these RV's X and Y are uncorrelated.

Q.8 A WSS random process $x(t)$ is applied to the input of an LTI system with impulse response

$$h(t) = 3e^{-2t}\,u(t)$$

Find the mean value of the output $y(t)$ of the system, if $E[x(t)] = 2$. Here E [·] denotes the expectation operator.

Solution:

The output $y(t)$ is the convolution of the input $x(t)$ and the impulse response $h(t)$.

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\cdot d\tau$$

$$\therefore \quad E[y(t)] = \int_{-\infty}^{\infty} h(\tau)\cdot E[x(t-\tau)]\cdot d\tau$$

$$E[y(t)] = H(0)\times E[x(t)]$$

$$\boxed{E[y(t)] = E[x(t)]\cdot H(0)}$$

where, $H(0) = H(\omega)\big|_{\omega=0}$ and $H(\omega)$ = Fourier transform of $h(t)$

Given $E[x(t)] = 2$, $\quad h(t) = 3e^{-2t}u(t)$

Taking Fourier transform, $\quad H(\omega) = \dfrac{3}{2+j\omega} \Rightarrow H(0) = \dfrac{3}{2}$

$$E[y(t)] = 2\times\frac{3}{2} = 3$$

Q.9 Suppose that two signals $s_1(t)$ and $s_2(t)$ are orthogonal over the interval $(0, T)$. A sample function $n(t)$ of a zero-mean white noise process is correlated with $s_1(t)$ and $s_2(t)$ separately, to yield the following variables:

$$n_1 = \int_0^T s_1(t)\,n(t)\,dt \quad \text{and} \quad n_2 = \int_0^T s_2(t)\,n(t)\,dt$$

Prove that n_1 and n_2 are orthogonal.

Solution:

$$E[n_1 n_2] = E\left[\int_0^T s_1(u)n(u)du \int_0^T s_2(v)n(v)dv\right]$$

$$= \int_0^T\int_0^T s_1(u)s_2(v)E[n(u)n(v)]\,du\,dv$$

$n(t)$ is a white noise process.

So, $\quad R_N(\tau) = \dfrac{N_0}{2}\delta(\tau)$

$$E[n(u)\,n(v)] = \frac{N_0}{2}\delta(u-v)$$

Hence, $\quad E[n_1 n_2] = \dfrac{N_0}{2}\displaystyle\int_0^T\int_0^T s_1(u)s_2(v)\delta(u-v)\,du\,dv$

$$= \frac{N_0}{2}\int_0^T s_1(u)s_2(u)\,du$$

$$= 0 \qquad \because s_1(t) \text{ and } s_2(t) \text{ are orthogonal over } (0, T)$$

$E[n_1 n_2] = 0$. So, n_1 and n_2 are also orthogonal.