POSTAL Book Package 2023

ESE

Electronics Engineering

Conventional Practice Sets

Electromagnetics

Contents

MADE EASY Publications

1 CHAPTER

Vector Analysis

Q.1 For a vector field $\vec{A}$, show explicitly that $\nabla \cdot \nabla \times \vec{A} = 0$; that is, the divergence of the curl of any vector field is zero.

Solution:

$$\nabla \cdot \nabla \times \vec{A} = \frac{\partial}{\partial x}\cdot \hat{a}_x + \frac{\partial}{\partial y}\cdot \hat{a}_y + \frac{\partial}{\partial z}\cdot \hat{a}_z \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\cdot \hat{a}_x + \frac{\partial}{\partial y}\cdot \hat{a}_y + \frac{\partial}{\partial z}\cdot \hat{a}_z \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{a}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{a}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{a}_z\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_x}{\partial y \partial z} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0$$

Because $\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x}$ and so on.

Q.2 If $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$, then show that the vector $\vec{A} = xz\,\hat{a}_x + z^2\,\hat{a}_y + yz\hat{a}_z$ satisfy this vector identity.

Solution:

We have to verify,

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \qquad \text{...(i)}$$

taking L.H.S

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & z^2 & yz \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x} z^2 - \frac{\partial}{\partial y} xz\right)\hat{a}_z + \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(z^2)\right)\hat{a}_x - \left(\frac{\partial}{\partial x} yz - \frac{\partial}{\partial z} xz\right)\hat{a}_y$$

$$= (z - 2z)\,\hat{a}_x + x\hat{a}_y = -z\hat{a}_x + x\hat{a}_y \qquad \text{...(ii)}$$

$$\therefore \quad \nabla\times(\nabla\times\vec{A}) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix}$$

$$= 0\cdot\hat{a}_x - (+1)\hat{a}_y + \hat{a}_z = -\hat{a}_y + \hat{a}_z \quad \text{...(iii)}$$

now considering R.H.S

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z$$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}yz = z + y$$

$$\nabla(\nabla\cdot\vec{A}) = \frac{\partial}{\partial x}(\nabla\cdot\vec{A})\hat{a}_x + \frac{\partial}{\partial y}(\nabla\cdot\vec{A})\hat{a}_y + \frac{\partial}{\partial z}(\nabla\cdot\vec{A})\hat{a}_z$$

$$\therefore \quad = \frac{\partial}{\partial x}(z+y)\hat{a}_x + \frac{\partial}{\partial y}(z+y)\hat{a}_y + \frac{\partial}{\partial z}(z+y)\hat{a}_z$$

$$= \hat{a}_y + \hat{a}_z \quad \text{...(iv)}$$

$$\nabla^2\vec{A} = \nabla^2 A_x\hat{a}_x + \nabla^2 A_y\hat{a}_y + \nabla^2 A_z\hat{a}_z$$

and

$$\nabla^2\vec{A} = \left[\frac{\partial^2}{\partial x^2}(xz) + \frac{\partial^2}{\partial y^2}(xz) + \frac{\partial^2}{\partial z^2}(xz)\right]\hat{a}_x$$

$$+\left[\frac{\partial^2}{\partial x^2}(z)^2 + \frac{\partial^2}{\partial y^2}(z)^2 + \frac{\partial^2}{\partial z^2}(z)^2\right]\hat{a}_y$$

$$+\left[\frac{\partial^2}{\partial x^2}(yz) + \frac{\partial^2}{\partial y^2}(yz) + \frac{\partial^2}{\partial z^2}(yz)\right]\hat{a}_z$$

$$\nabla^2\vec{A} = 2\hat{a}_y \quad \text{...(v)}$$

From equation (iv) and (v), we get,

$$\nabla\cdot(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \hat{a}_y + \hat{a}_z - 2\hat{a}_y = \hat{a}_z - \hat{a}_y \quad \text{...(vi)}$$

$\therefore$ From equation (iii) and (vi),

$$\text{RHS} = \text{LHS} = \hat{a}_z - \hat{a}_y$$

Hence Proved.

Q.3 If $\vec{F} = x^2y\hat{a}_x - 2z\hat{a}_y + (3z^2 + xy)\hat{a}_z$, find $\nabla\times[\nabla.(\nabla.\vec{F})]$.

Solution:

Given $\vec{F} = x^2y\hat{a}_x - 2z\hat{a}_y + (3z^2 + xy)\hat{a}_z$

Let us calculate $\nabla\times[\nabla.(\nabla.\vec{F})]$ step by step.

$$\nabla.\vec{F} = \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(3z^2 + xy)$$

$$= 2xy + 6z$$

$$\nabla.(\nabla.\vec{F}) = \nabla\,.\,(2xy + 6z)$$

$$= \frac{\partial}{\partial x}(2xy+6z)\hat{a}_x + \frac{\partial}{\partial y}(2xy+6z)\hat{a}_y + \frac{\partial}{\partial z}(2xy+6z)\hat{a}_z$$

$$= 2y\hat{a}_x + 2x\hat{a}_y + 6\hat{a}_z$$

$$\nabla\times[\nabla.(\nabla.\vec{F})] = \nabla\times(2y\hat{a}_x + 2x\hat{a}_y + 6\hat{a}_z)$$

$$= \begin{bmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 2x & 6 \end{bmatrix}$$

$$= \left(\frac{\partial(6)}{\partial y} - \frac{\partial(2x)}{\partial z}\right)\hat{a}_x - \left(\frac{\partial(6)}{\partial x} - \frac{\partial(2y)}{\partial z}\right)\hat{a}_y + \left(\frac{\partial(2x)}{\partial x} - \frac{\partial(2y)}{\partial y}\right)\hat{a}_z$$

$$= 0\hat{a}_x - 0\hat{a}_y + (2-2)\hat{a}_z = 0$$

Note that $\nabla\times\nabla\cdot\vec{F} = 0$

Q.4 E and F are vector fields given by $E = 2x\,\hat{a}_x + \hat{a}_y + yz\hat{a}_z$ and $F = xy\,\hat{a}_x - y^2\hat{a}_y + xy\,z\hat{a}_z$. Determine

(a) $|E|$ at (1, 2, 3)

(b) The component of E along F at (1, 2, 3)

(c) A vector perpendicular to both E and F at (0, 1, –3) whose magnitude is unity.

Solution:

(a) $\vec{E} = 2x\,\hat{a}_x + \hat{a}_y + yz\,\hat{a}_z$

At point (1, 2, 3) $\Rightarrow$ $\vec{E} = 2\hat{a}_x + \hat{a}_y + 6\hat{a}_z$

$$|\vec{E}| = \sqrt{2^2 + 1^2 + 6^2} = \sqrt{41} = 6.403$$

(b) $\vec{F} = xy\,\hat{a}_x - y^2\,\hat{a}_y + xyz\,\hat{a}_z$

At (1, 2, 3), $\vec{F} = 2\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z$

$\therefore$ The component of $\vec{E}$ along $\vec{F}$

$$E_F = (\vec{E}\cdot\hat{a}_F)\hat{a}_F \qquad \left[\hat{a}_F = \frac{\vec{F}}{|\vec{F}|}\right]$$

$$= \frac{(\vec{E}\cdot\vec{F})}{|\vec{F}|^2}\vec{F} = \frac{36}{56}(2\hat{a}_x - 4\hat{a}_y + 6\hat{a}_z) = 1.286\hat{a}_x - 2.571\hat{a}_y + 3.857\hat{a}_z$$

(c) At (0, 1, –3) $\vec{E} = 0\hat{a}_x + \hat{a}_y - 3\hat{a}_z$

$\vec{F} = 0\hat{a}_x - \hat{a}_y + 0\hat{a}_z$

$$E\times\vec{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = -3\hat{a}_x + 0\hat{a}_y + 0\hat{a}_z$$

$$a_{E\times F} = \pm\frac{\vec{E}\times\vec{F}}{|\vec{E}\times\vec{F}|} = \pm\hat{a}_x$$

Q.5 Determine the divergence of the vector fields.

(i) $\vec{P} = x^2yz\hat{a}_x + xz\hat{a}_z$ where $\hat{a}_x$ and $\hat{a}_z$ are unit vectors along x and z axis respectively.

(ii) $\vec{Q} = \rho\sin\phi\hat{a}_\rho + \rho^2 z\hat{a}_\phi + z\cos\phi\hat{a}_z$ where $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are unit vectors in cylindrical co-ordinate system.

Solution:

(i)
$$\vec{P} = x^2yz\hat{a}_x + xz\hat{a}_z$$
$$\text{Divergence of } \vec{P} = \nabla\cdot\vec{P}$$
$$= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)$$
$$= 2xyz + x$$

(ii)
$$\vec{Q} = \rho\sin\phi\hat{a}_\rho + \rho^2 z\hat{a}_\phi + z\cos\phi\hat{a}_z$$
$$\nabla\cdot\vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\cdot\rho\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)$$
$$= \frac{2\rho\sin\phi}{\rho} + \frac{1}{\rho}\times 0 + \cos\phi = 2\sin\phi + \cos\phi$$

Q.6 Given $\vec{V} = x\cos^2 y\hat{i} + x^2e^z\hat{j} + z\sin^2 y\,\hat{k}$ and S the surface of unit cube with one corner at the origin and edges parallel to the coordinate axis. Calculate the value of the integral $\iint_C \vec{V}\cdot\hat{n}\,dS$.

Solution:

Using divergence theorem,
$$\oiint \vec{V}\cdot\vec{ds} = \iiint(\nabla\cdot\vec{V})\,dv$$
$$\nabla\cdot\vec{V} = \frac{\partial}{\partial x}(x\cos^2 y) + \frac{\partial}{\partial y}(x^2e^z) + \frac{\partial}{\partial z}(z\sin^2 y)$$
$$= \cos^2 y + \sin^2 y$$
$$= 1$$
$$\therefore \quad \iiint(\nabla\cdot\vec{V})\cdot dV = \iiint dV = \text{Volume of unit cube} = 1$$

Q.7 Let $\vec{H} = 5\rho\sin\phi\,\hat{a}_\rho - \rho z\cos\phi\hat{a}_\phi + 2\rho\hat{a}_z$ A/m. At point $P(2, 30°, -1)$ find:

(a) a unit vector along $\vec{H}$.

(b) the component of $\vec{H}$ parallel to $\hat{a}_x$.

(c) the component of $\vec{H}$ normal to $\rho = 2$.

(d) the component of $\vec{H}$ tangential to $\phi = 30°$.

Solution:

At P,
$$\rho = 2,\ \phi = 30°,\ z = -1$$
$$\vec{H} = 10\sin 30\,\hat{a}_\rho + 2\cos 30°\hat{a}_\phi + 4\hat{a}_z$$
$$= 5\hat{a}_\rho + 1.732\,\hat{a}_\phi + 4\hat{a}_z \text{ A/m}$$

(a) Unit vector along $\vec{H}$,

$$\hat{a}_H = \frac{5\hat{a}_\rho + 1.732\,\hat{a}_\phi + 4\hat{a}_z}{\sqrt{5^2 + 1.732^2 + 4^2}} = 0.7538\,\hat{a}_\rho + 0.2611\,\hat{a}_\phi + 0.603\,\hat{a}_z$$

(b) The relation between Cartesian coordinates (A_x, A_y, A_z) and cylindrical coordinates (A_ρ, A_ϕ, A_z) is given as:

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

and

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$H_x = H_\rho \cos\phi - H_\phi \sin\phi$$
$$= 5\rho \sin\phi \cos\phi - \rho z \cos\phi \sin\phi$$

At P, $\rho = 5, \quad \phi = 30°, z = -1$

$$H_x = H_x\,\hat{a}_x = (25\sin 30° \cos 30° + 5\sin 30° \cos 30°)\,\hat{a}_x$$
$$= 13\hat{a}_x \text{ A/m}$$

(c) Normal to $\rho = 2$ is $\vec{H}_n = \vec{H}_P\,\hat{a}_p$

i.e. $\vec{H}_n = 0.7538\,\hat{a}_\rho$ A/m

(d) Tangential to $\phi = 30°$

$$H_t = H_\rho \cdot \hat{a}_p + H_z\,\hat{a}_z$$
$$= 0.7538\,\hat{a}_p + 0.603\,\hat{a}_z \text{ A/m}$$

Q.8 Find the rate at which the scalar function, $V = r^2 \sin 2\phi$, in cylindrical co-ordinates, increases in the direction of the vector $A = \hat{a}_r + \hat{a}_\phi$ at the point having co-ordinates $(2, \pi/4, 0)$.

Solution:

As we know that, Gradient is a vector that represents both the magnitude and the direction of maximum space rate of the increase of the scalar function i.e.

$$\text{grad } V = \nabla V = \frac{dV}{dn}\hat{a}_n \qquad \text{...(i)}$$

But in cylindrical coordinate system, the grad V can be defined as,

$$\nabla V = \frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z \qquad \text{...(ii)}$$

For the case under consideration, the quantity required is,

$$\nabla V \cdot \hat{a}_A = \nabla V \cdot \frac{\vec{A}}{|\vec{A}|} \quad \text{at } (2, \pi/4, 0)$$

From equation (ii) we have,

$$\nabla V = \frac{\partial}{\partial r}(r^2 \sin 2\phi)\hat{a}_r + \frac{1}{r}\frac{\partial}{\partial \phi}(r^2 \sin 2\phi)\hat{a}_\phi + \frac{\partial}{\partial z}(r^2 \sin 2\phi)\hat{a}_z.$$

or, $\nabla V = 2r \sin 2\phi\ \hat{a}_r + 2r \cos 2\phi\ \hat{a}_\phi$

Now, $$\nabla V \cdot \vec{A} = (2r\sin 2\phi\,\hat{a}_r + 2r\cos 2\phi\,\hat{a}_\phi)\cdot(\hat{a}_r + \hat{a}_\phi)$$

or, $$\nabla V \cdot \vec{A} = 2r\sin 2\phi + 2r\cos 2\phi \quad \text{...(iii)}$$

Also, $$|\vec{A}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$\therefore$ $$\nabla V \cdot \hat{a}_A = \frac{\nabla V \cdot \vec{A}}{|\vec{A}|} = \frac{1}{\sqrt{2}}[2r\sin 2\phi + 2r\cos 2\phi]$$

$$= (\sqrt{2}\,r\sin 2\phi + \sqrt{2}\,r\cos 2\phi)$$

Now, $$(\nabla V \cdot \hat{a}_A)_{at(2,\pi/4,0)} = \sqrt{2}\times 2\times \sin\left(2\times\frac{\pi}{4}\right) + \sqrt{2}\times 2\times\cos\left(2\times\frac{\pi}{4}\right) = 2\sqrt{2}$$

Q.9 Write down the Divergence theorem. An electric field at point P, expressed in cylindrical co-ordinate system is given by,

$$\vec{E} = 3r^2\sin\phi\,\hat{a}_r + 2r^2\cos\phi\,\hat{a}_\phi$$

Find the value of divergence of the field if the location of point 'P' is given by (9, 9, 9) in Cartesian co-ordinate system.

Solution:

Divergence Theorem: The divergence theorem states that, the volume integral of the divergence of a vector field 'A', taken over any volume V is equal to the surface integral of A, taken over the closed surface that is

$$\int_V (\nabla\cdot\vec{A})\,dV = \oint_S \vec{A}\cdot ds$$

Given that,

$$\vec{E} = 3r^2\sin\phi\,\hat{a}_r + 2r^2\cos\phi\,\hat{a}_\phi$$

In cylindrical coordinate system,

$$\nabla\cdot\vec{E} = \frac{1}{r}\frac{\partial}{\partial r}(rEr) + \frac{1}{r}\frac{\partial E\phi}{\partial\phi} + \frac{\partial Ez}{\partial z}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(3r^3\sin\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(2r^2\cos\phi)$$

$$= \frac{1}{r}(9r^2\sin\phi) + \frac{1}{r}(2r^2(-\sin\phi))$$

$$= 9r\sin\phi - 2r\sin\phi$$

$$\nabla\cdot\vec{E} = 7r\sin\phi$$

Converting into Cartesian co-ordinates

$$r = \sqrt{x^2+y^2}$$

and $$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$$

$\therefore$ $$\nabla\cdot\vec{E} = 7\sqrt{x^2+y^2}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = 7\sqrt{9^2+9^2}\cdot\frac{9}{\sqrt{9^2+9^2}} = 7\times 9 = 63$$

$$\nabla\cdot\vec{E} = 63$$