Production & Industrial Engineering

Operations Research and Operations Management

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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Corporate Office: 44-A/4, Kalu Sarai (Near Hauz Khas Metro Station), New Delhi-110016
E-mail: infomep@madeeasy.in
Contact: 011-45124660, 8860378007
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CHAPTER 01

Linear Programming and Its Applications

INTRODUCTION

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions. Linear programming technique may be used for solving broad range of problems arising in business, government, industry, hospitals, libraries, etc. Whenever we want to allocate the available limited resources for various competing activities for achieving our desired objective, the technique that helps us is Linear Programming. As a decision making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more. In this chapter, let us discuss about various types of linear programming models.

1.1 Properties of Linear Programming Model

Any linear programming model (problem) must have the following properties:

(a) The relationship between variables and constraints must be linear.

(b) The model must have an objective function.

(c) The model must have structural constraints.

(d) The model must have non-negativity constraint.

1.2 Standard Form of LPP

A general mathematical way of representing a Linear Programming Problem (L.P.P.) is as given below :

$Z = c_1x_1 + c_2x_2 + \ldots\ldots c_nx_n$ subjects to the conditions, ⟶ OBJECTIVE FUNCTION

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{ij}x_j + \ldots\ldots + \ldots\ldots a_{1n}x_n \ (\geq, =, \leq)\ b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2j}x_j + \ldots\ldots + a_{2n}x_n \ (\geq, =, \leq)\ b_2$

....

....

$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mj}x_j \ldots\ldots + a_{mn}x_n \ (\geq, =, \leq)\ b_m$

(Structural Constraints)

and all x_j are = 0 ⟶ NON NEGETIVITY CONSTRAINT

where $j = 1, 2, 3, \ldots\ldots, n$

where,

$$x_1 \geq 0, x_2 \geq 0 \ldots x_n \geq 0, x_{n+1} \geq 0 \ldots x_{n+m} \geq 0$$

where $x_1, x_2, \ldots x_n$ are called **decision variables**.

$x_{n+1}, x_{n+2} \dots x_{n+m}$
are called slack variable.

c_1, c_2, ... c_n are called cost factors.

- Coefficient of slack variables $x_{n+1}, x_{n+2} \dots x_{n+m}$ in the objective function are assumed to be zero.
- Since in case of ($\geq$ 0) constraints, the subtracted variables represents the surplus of the left side over the right side, it is common to refer to it as surplus variables.
- Solution to LPP: Any set $x\{x_1, x_2, \dots x_{n+m}\}$ of variables is called as a solution to LPP, it is satisfy the set of constraints (2) only.

The steps for formulating the linear programming are:

1. Identify the unknown decision variables to be determined and assign symbols to them.
2. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.
3. Identify the objective or aim and represent it also as a linear function of decision variables.

1.3 Basic Assumptions

The following are some important assumptions made in formulating a linear programming model :

1. It is assumed that the decision maker here is completely certain (i.e., deterministic conditions) regarding all aspects of the situation, i.e., availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.
2. It is assumed that the relationship between variables in the problem and the resources available i.e., constraints of the problem exhibits linearity. Here the term linearity implies proportionality and additivity. This assumption is very useful as it simplifies modeling of the problem.
3. We assume here fixed technology. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.
4. It is assumed that the profit contribution of a product remains constant, irrespective of level of production and sales.
5. It is assumed that the decision variables are continuous. It means that the companies manufacture products in fractional units. For example, company manufacture 2.5 vehicles, 3.2 barrels of oil etc. This is referred too as the assumption of divisibility.
6. It is assumed that only one decision is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a single stage decision problem. (Note: Dynamic Programming problem is a multistage decision problem).
7. All variables are restricted to nonnegative values (i.e., their numerical value will be ≥ 0).

1.4 Steps to Formulate LPP Model

The steps for formulating the linear programming are:

1. Identify the unknown decision variables to be determined and assign symbols to them.
2. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.
3. Identify the objective or aim and represent it also as a linear function of decision variables.

We can further categories the problem into maximization and minimization model.

1.4.1 Maximization Model

Herein, our aim is to maximize the objective function with all the variables subjected to given constraint. Following example illustrates maximization model of a linear programming problem.

Example 1.1 A retail store stocks two types of shirts *A* and *B*. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type *A* and a maximum of 300 shirts of type *B*. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type *A* shirt fetches a profit of Rs. 2/- per unit and type *B* a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Solution:

Here shirts *A* and *B* are problem variables. Let the store stock 'a' units of *A* and 'b' units of *B*. As the profit contribution of *A* and *B* are Rs. 2/- and Rs. 5/- respectively, objective function is: Maximize $Z = 2a + 5b$ subjected to condition (s.t.)

Structural constraints are, stores can sell 400 units of shirt *A* and 300 units of shirt *B* and the storage capacity of both put together is 600 units. Hence the structural constraints are: $1a + 0b \geq 400$ and $0a + 1b \leq 300$ for sales capacity and $1a + 1b \leq 600$ for storage capacity. And non-negativity constraint is both a and b are ≥ 0. Hence the model is:

$$\text{Maximize } Z = 2a + 5b \text{ s.t.}$$
$$1a + 0b \leq 400$$
$$0a + 1b \leq 300$$
$$1a + 1b \leq 600 \text{ and both } a \text{ and } b \text{ are } \geq 0.$$

1.4.2 Minimization Model

Herein, our aim is to maximize the objective function with all the variables subjected to given constraint. Following example illustrates maximization model of a linear programming problem.

Example 1.2 A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and tonic Y, which are having vitamin A, and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin Daily. The cost of tonics *X* and *Y* and the proportion of vitamin A and D that present in *X* and *Y* are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	Tonics		Daily requirement in units
	X	*Y*	
A	2	4	40
D	3	2	50
Cost in Rs. per unit	5	3	

Solution:

Let patient purchase x units of *X* and y units of *Y*.

Objective function: Minimize $Z = 5x + 3y$

Inequality for vitamin A is $2x + 4y \geq 40$ (Here at least word indicates that the patient can consume more than 40 units but not less than 40 units of vitamin A daily).

Similarly, the inequality for vitamin *D* is $3x + 2y \geq 50$.

For non-negativity constraint the patient cannot consume negative units. Hence both x and y must be ≥ 0.

Now the l.p.p. model for the problem is :

$$\text{Minimize Z} = 5x + 3y \text{ s.t.}$$
$$2x + 4y \geq 40$$
$$3x + 2y \geq 50 \text{ and}$$

Both x and y are ≥ 0.

1.11 Simulation

Simulation is the numerical technique for conducting experiments that involve certain types of mathematical and logical relationship necessary to describe the behaviour and structure of a complex real world system over extended period of time.

In other words we can say that simulation is an especially valuable tool in a situation where the mathematics needed to describe a system realistically is too complex to yield analytically solution.

1.11.1 Rationale

With the help of simulation one can introduce the constant and variables required to the problem, setup the possible course of action and establish criteria which act as measure of effectiveness.

Following list justifies the rationale.

(i) Simulation is the only method available because the actual environment is difficult to observe in reality.

(ii) It may not be possible to develop an analytical solution of the problem.

(iii) Actual observation of a system is prohibitively expensive and time consuming.

(iv) There is not sufficient time available to allow the system to operate extensively.

(v) Actual operation and observation of a system may be too disruptive.

(vi) It provides a trial and error movement towards the optimal solution. The decision matter selects an alternative experience as the effect of the selection and then improves the selection. This way the selection is adjusted until approximates the optimal solution.

1.11.2 Monte Carlo Simulation

It is a simulation technique in which statistical distribution functions are created by using a series or random numbers. This approach has the ability to develop many months or year date in matter of a few minutes of digital computer.

This method is used to solve problems which cannot be adequately represented by the mathematical models or where solution of model is not possible by analytical method.

Procedure of Simulation

Basic Steps

1. Identify the problem
2. Construct a model of the given problem
3. Testing the model
4. Identify the collecting date
5. Running the simulation process
6. Making change is required in the model or parameters
7. Rerunning the solution to test new solution,

Advantages of Simulation

(i) The study of very complicated system can be done with the help of simulation.

(ii) We can investigate the consequences for a system of possible changes in parameter in terms of the model by simulation.

(iii) The knowledge of a system obtained in designing and constructing the simulation is very valuable.

(iv) In business games, case studies etc. it is teaching aid.

(v) The simulation of complicated system helps us to locate which variable have the important influence; on system performance.

(vi) This can be used to experiment unfamiliar system to prepare routine and extreme eventualities.

(vii) Simulation methods are easier to apply than pure analytical method. When all else tails – simulation. The last resort is simulation when all else fail, so should we call it ZIMULATION.

Example 1.21 Solve the following linear programming problem :

Maximize $z = 3x_1 + 2x_2 + 5x_3$

subject to

$x_1 + 2x_2 + x_3 \le 430$ $\quad 3x_1 + 2x_3 \le 460$

$x_1 + x_2 \le 420$ $\quad x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

Solution:

Maximize, $z = 3x_1 + 2x_2 + 5x_3$

subject to
$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + x_2 \le 420$$
$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$$

Standard Form:

$$Z = 3x_1 + 2x_2 + 5x_3 + 0.S_1 + 0.S_2 + 0.S_3$$

Subject to,
$$x_1 + 2x_2 + x_3 + S_1 = 430$$
$$3x_1 + 2x_3 + S_2 = 460$$
$$x_1 + x_2 + S_3 = 420$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \ge 0$$

Initial solution : $S_1 = 430, S_2 = 460, S_3 = 420$

θ_1	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_i	$\theta_1 = b_i / a_j$
0	S_1	1	2	1	1	0	0	430	430
0	S_2	3	0	[2]	0	1	0	460	230 ←
0	S_3	1	1	0	0	0	1	420	∞
	C_j	3	2	5					
	Z_j	0	0	0					
	Δ_j	3	2	5 ↑					

Ist iteration :

e_j	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_i	$\theta_1 = b_i / a_j$
0	S_1	– 0.5	[2]	0	1	– 0.5	0	200	100 ←
5	x_3	1.5	0	1	0	0.5	0	230	∞
0	S_3	1	1	0	0	0	1	420	420
	C_j	3	2	5					
	Z_j	7.5	0	5					
	Δ_j	– 4.5	2 ↑	0					

$$S_1 = 100, x_3 = \infty, S_3 = 420$$

IInd iteration :

e_j	Basis	x_1	x_2	x_3	S_1	S_2	S_3	b_i	θ_1
2	x_2	– 0.25	1	0	0.5	– 0.25	0	100	
5	x_3	1.5	0	1	0	0.5	0	230	
0	S_3	1.25	0	0	– 0.5	0.25	1	320	
	C_j	3	2	5					
	Z_j	7	2	5					
	Δ_j	– 4	0	0					

This is an optimal solution as all values in Δ_j are negative or zero.

$$x_1 = 0;\ x_2 = 100;\ x_3 = 230;\ S_3 = 320$$

$$Z = 3x_1 + 2x_2 + 5x_3$$

$$= 2 \times 100 + 5 \times 230 = 1350$$

Example 1.22 XYZ Corporation produces 2 products. Unit profit for product *A* is Rs. 120/- and for product *B* is Rs. 100. Each must pass through 2 machines *P* and *Q*. Product *A* requires 20 minutes on machine *P* and 16 minutes on machine *Q*. Product *B* requires 40 minutes on machine *P* and 10 minutes on machine *Q*. Machine *P* is available 400 minutes a day, while *Q* is available 160 minutes a day. Due to operational set-up, the company must produce at least 2 units of product *A* and 5 of product *B* each day. Units that are not completed in a given day are finished in the next day : that is portion of a product can be produced in the daily plan.

(i) Formulate the Linear Programming Problem.

(ii) Solve graphically showing all equations to find the most profitable production plan.

(iii) How should the resources be allocated?

Solution:

Product	Machines	
	P	*Q*
A	20	16
B	40	10
Daily available	400	160

Let x be the number of quantities of product *A* and y be the number of quantities of product *B* produced.

Objective Function:

$$\text{Maximize } Z = 120x + 100y$$

Constraints:

$$20x + 40y \leq 400 \quad \text{...(1)}$$
$$16x + 10y \leq 160 \quad \text{...(2)}$$

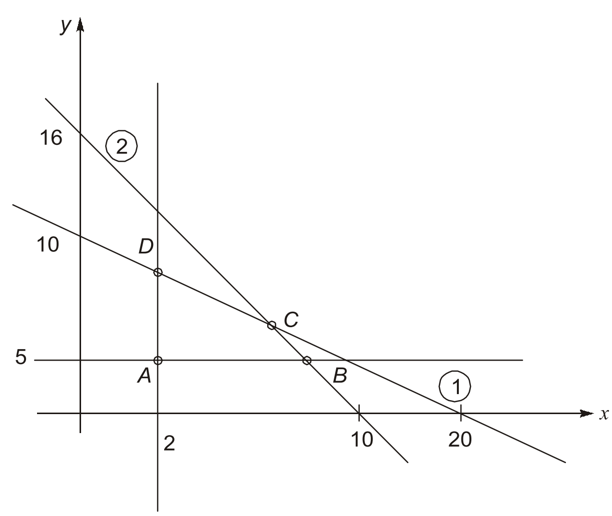
and $$x \geq 2,\ y \geq 5 \quad \text{...(3)}$$

Therefore, $$\frac{x}{20} + \frac{y}{10} \leq 1 \quad \text{...(1)}$$

$$\frac{x}{10} + \frac{y}{16} \leq 1 \quad \text{...(2)}$$

$$x \geq 2 \text{ and } y \geq 5 \quad \text{...(3)}$$

$A \rightarrow (2, 5)$; $B \rightarrow (6.875, 5)$; $C \rightarrow (5.45, 7.27)$; $D \rightarrow (2, 9)$



$$Z_{(A)} = 120 \times 2 + 100 \times 5 = 740$$
$$Z_{(B)} = 120 \times 6.875 + 100 \times 5 = 1325$$
$$Z_{(C)} = 120 \times 5.45 + 100 \times 7.27 = 1381 \leftarrow \text{(maximum)}$$
$$Z_{(D)} = 120 \times 2 + 100 \times 9 = 1140$$

Therefore daily production should be

Product $A \rightarrow 5.45$

Product $B \rightarrow 7.25$

and daily profit (Z) = Rs. 1381

Example 1.23 A company is manufacturing two different products *A* and *B*. Each product is to be processed in three departments-casting, machining and finally inspection. The capacity of 3 departments is limited to 35 hours, 32 hours and 24 hours per week respectively. Product *A* requires 7 hours in casting department, 8 hours in machining shop and 4 hours in inspection whereas product B requires 5 hours, 4 hours and 6 hours respectively in respective shops. The profit contribution for a unit product of *A* and *B* is Rs. 30/- and Rs. 40/- respectively.

(i) Formulate the problem.

(ii) Find out the optimal quantities of products *A* and *B*.

Solution :

Let x and y be the optimal quantities of products *A* and *B* produced respectively.

	Casting	Machining	Inspection
Product *A*	7	8	4
Product *B*	5	4	6
Capacity	35	32	24

Objective function :

$$\text{Maximize } Z = 30x + 40y$$

Constraints :

$$7x + 5y \le 35$$

$$\frac{x}{5} + \frac{y}{7} \le 1 \quad \text{...(1)}$$

$$8x + 4y \le 32$$

$$\frac{x}{4} + \frac{y}{8} \le 1 \quad \text{...(2)}$$

$$4x + 6y \le 24$$

$$\frac{x}{6} + \frac{y}{4} \le 1 \quad \text{...(3)}$$

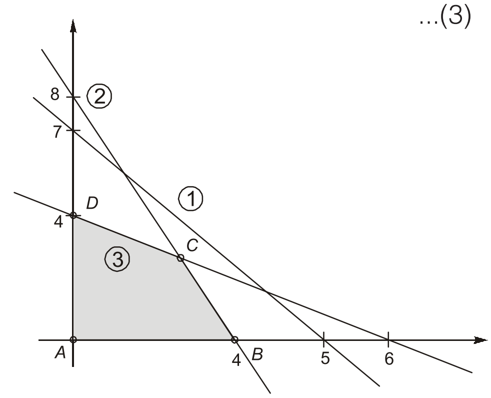
and $x \ge 0,\ y \ge 0$

$A \to (0, 0) \Rightarrow Z_A = 0$

$B \to (4, 0) \Rightarrow Z_B = 120$

$C \to (3, 2) \Rightarrow Z_C = 170$

$D \to (0, 4) \Rightarrow Z_D = 160$



Since, Z_C is maximum, optimal production is :

Quantity of Product *A* = 3

Quantity of Product *B* = 2

Example 1.24 The standard weight of a block is 5 kg and it contains sand and cement. The sand costs Rs. 0.05 per kg and cement costs Rs. 0.08 per kg. The block to be strong must not contain more than 4 kg of sand and must contain less than 2 kg of cement. What is the minimum cost of the block?

Solution:

Let each block contains x kg sand y kg cement

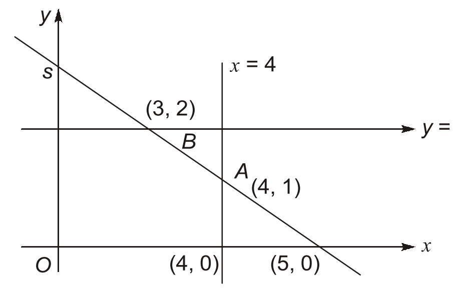
Cost of each block, $Z = 0.05x + 0.08y$

Objective Minimize, $Z = 0.05x + 0.08y$

Subject to, $x + y = 5$ kg

and $x \le 4,\ y \le 2$

and $x,\ y \ge 0$



$$Z_A = 0.05 \times 4 + 0.08 \times 1$$
$$= \text{Rs. } 0.28 \leftarrow \text{(Minimum)}$$
$$Z_B = 0.05 \times 3 + 0.08 \times 2 = \text{Rs. } 0.31$$

Minimum cost of Ihe block is Rs. 0.28

Example 1.25 Using LPP Graphical method, find the maximum value of $Z = 5x_1 + 3x_2$ subject to constraints:

$3x_1 + 5x_2 \le 15$, $5x_1 + 2x_2 \le 10$ and $x_1, x_2 \le 0$

Find the values of x_1 and x_2 for the condition for maximization.

Solution :

Maximum $Z = 5x_1 + 3x_2$

Subject to constraints: $3x_1 + 5x_2 \le 15$

$$\frac{x_1}{5} + \frac{x_2}{3} \le 1 \quad \text{...(1)}$$

$$5x_1 + 2x_2 \le 10$$

$$\frac{x_1}{2} + \frac{x_2}{5} \le 1 \quad \text{...(2)}$$

$$x_1 \ge 0 \text{ and } x_2 \ge 0 \quad \text{...(3)}$$

$A \rightarrow (2, 0)$

$Z_A = 5 \times 2 = 10$

$B \rightarrow (1.05, 2.36)$

$Z_B = 5 \times 1.05 + 3 \times 2.36$

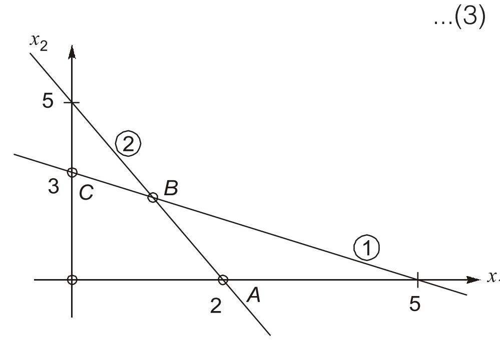
$= 12.33$

$C \rightarrow (0, 3)$

$Z_C = 3 \times 3 = 9$

for maximum Z, $x_1 = 1.05$

$x_2 = 2.26$ and $Z_{max} = 12.33$



Example 1.26 Solve the assignment problem represented by the matrix given below:

	a	b	c	d	e	f
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	‘32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Solution :

1. Row minima :

0	13	49	2	10	18
0	35	29	7	20	5
13	0	63	9	17	5
47	15	0	22	12	5
25	0	46	11	14	7
0	53	50	28	14	25

2. Column minima :

0	13	49	0	0	13
0	35	29	5	10	0
13	0	63	7	7	0
47	15	0	20	2	0
25	0	46	9	4	2
0	53	50	26	4	20

Now, all the zeroes can be covered by a minimum of 5 lines i.e., it is not equal to the number of rows or column. Hence optimal solution is not reached.

Examining rows successively until a row with exactly one unmarked zero is found and making there an assignment and leaving other zeros in that row or column unassigned.

~~0~~	13	49	[0]	~~0~~	13	
~~0~~	35	29	5	10	[0]	✓
13	~~0~~	63	7	7	~~0~~	✓
47	15	[0]	20	2	~~0~~	
25	[0]	46	9	4	2	✓
[0]	53	50	26	4	20	✓
✓	✓				✓	

As no. of allocations is 5 which is less than number of rows, optimal solution is not reached. Performing optimality test, we get

4	17	49	[0]	~~0~~	17
[0]	35	25	1	6	~~0~~
13	~~0~~	59	3	3	[0]
51	19	[0]	20	2	4
25	[0]	42	5	~~0~~	2
~~0~~	53	46	22	[0]	20

We get the optimal solution :

$B \rightarrow a;\ E \rightarrow b;\ D \rightarrow c;\ A \rightarrow d;\ F \rightarrow e;\ C \rightarrow f$