ESE GATE

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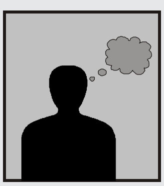
Workbook 2023



Detailed Explanations of Try Yourself *Questions*

Mechanical Engineering
Engineering Mathematics

MADE EASY
Publications

1 Linear Algebra

Detailed Explanation *of* Try Yourself Questions

T1 : Solution

(a)

With the given order we can say that order of matrices are as follows:

$$X^T \rightarrow 3 \times 4$$
$$Y \rightarrow 4 \times 3$$
$$X^T Y \rightarrow 3 \times 3$$
$$(X^T Y)^{-1} \rightarrow 3 \times 3$$
$$P \rightarrow 2 \times 3$$
$$P^T \rightarrow 3 \times 2$$
$$P(X^T Y)^{-1} P^T \rightarrow (2\times3)\,(3 \times 3)\,(3 \times 2) \rightarrow 2 \times 2$$
$$\therefore \qquad (P(X^T Y)^{-1} P^T)^T \rightarrow 2 \times 2$$

T2 : Solution

(b)

The matrix can be

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{and} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

Number of ways possible for one matrix = 4^4

$\Rightarrow$ Total number of matrix possible = 3×4^4

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

T3 : Solution

(c)

$$A\begin{bmatrix}2\\-1\\0\end{bmatrix}=\begin{bmatrix}2\\-1\\0\end{bmatrix},\ A\begin{bmatrix}-1\\2\\-1\end{bmatrix}=\begin{bmatrix}-2\\4\\-2\end{bmatrix},\ A\begin{bmatrix}0\\-1\\2\end{bmatrix}=\begin{bmatrix}0\\-3\\6\end{bmatrix}$$

$$Q = \begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 2\end{bmatrix}$$

Let, $$T_1 = \begin{bmatrix}2\\-1\\0\end{bmatrix},\ T_2 = \begin{bmatrix}-1\\2\\-1\end{bmatrix},\ T_3 = \begin{bmatrix}0\\-1\\2\end{bmatrix}$$

$$AT_1 = \begin{bmatrix}2\\-1\\0\end{bmatrix},\ AT_2 = \begin{bmatrix}-2\\4\\-2\end{bmatrix},\ AT_3 = \begin{bmatrix}0\\-3\\6\end{bmatrix}$$

$$A[T_1\ T_2\ T_3] = \begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 2\end{bmatrix}$$

$$[T_1\ T_2\ T_3] = \begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 2\end{bmatrix} = Q$$

$$A[T_1\ T_2\ T_3] = AQ = \begin{bmatrix}2 & -2 & 0\\-1 & 4 & -3\\0 & -2 & 6\end{bmatrix}$$

T4 : Solution

(c)

Method 1:

$$E = \begin{bmatrix}\cos & -\sin\theta & 0\\\sin\theta & \cos\theta & 0\\0 & 0 & 1\end{bmatrix}$$

and $$G = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

According to problem

$$E \times F = G$$

or $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence we see that product of $(E \times F)$ is unit matrix so F has to be the inverse of E.

$$F = E^{-1} = \frac{Adj(E)}{|E|}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Method 2:

An easier method for finding F is by multiplying E with each of the choices (a), (b), (c) and (d) and finding out which one gives the product as identity matrix G. Again the answer is (c).

T5 : Solution

(a)

For singularity of matrix $= \begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$\Rightarrow$ $8(0 - 12) - x(0 - 2 \times 12) = 0$

$\therefore$ $x = 4$

T6 : Solution

(b)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{(ad - bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}^{-1} = \frac{1}{\left[(3+2i)(3-2i)+i^2\right]}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

$$= \frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

T7 : Solution

(b)

Take the determinant of given matrix $|A|$

$= 2[2(4-1) - 1(2-1) + 1(1-2)] - 1[1(4-1) - 1(2-1) + 1(1-2)]$
$\quad +1[1(2-1) - 2(2-1) + 1(1-1)] - 1[1(1-2) - 2(1-2) + 1(1-1)]$

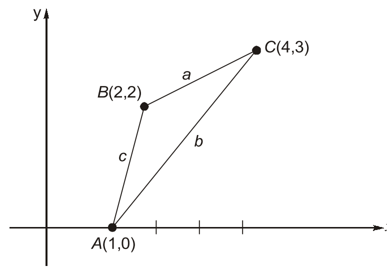
$= 2[6 - 1 - 1] - 1[3 - 1 - 1] + 1[1 - 2 + 0] - 1[-1 + 2 + 0]$

$= 2(4) - 1(1) + 1(-1) - 1(1) = 8 - 1 - 1 - 1$

$= 5$

T8 : Solution

(a)



Area of the triangle

$$= \frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \right|$$

$$= \frac{1}{2}\left| 1(2-3) + 2(3-0) + 4(0-2) \right| = \frac{1}{2}\left| -1 + 6 - 8 \right|$$

$$= \frac{3}{2}$$

T9 : Solution

(c)

Consider first 3×3 minors, since maximum possible rank is 3

$$\begin{vmatrix} 4 & 2 & 1 \\ 6 & 3 & 4 \\ 2 & 1 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 & 3 \\ 3 & 4 & 7 \\ 1 & 0 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 4 & 1 & 3 \\ 6 & 4 & 7 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

and $$\begin{vmatrix} 4 & 2 & 3 \\ 6 & 3 & 7 \\ 2 & 1 & 1 \end{vmatrix} = 0$$

Since all 3×3 minors are zero, now try 2×2 minors.

$$\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5 \neq 0$$

So, rank $= 2$

T10 : Solution

Given : $A = a_{ij}$

where, $a_{ij} = 5 \forall i, j$

So, $$A = \begin{bmatrix} 5 & 5 & 5 & \cdots & \cdots & \cdots \\ 5 & 5 & 5 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

In a matrix if all the rows or columns are proportional or identical then its rank is always 1.

Hence, Rank = 1

T11 : Solution

Given matrix is 4×4 skew-symmetric matrix

$\therefore$ It is non-singular

Therefore rank of $A = 4$.

T12 : Solution

(a)

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right]$$

Using Gauss-elimination method we get

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 3 & 3 & 9 \\ 1 & 2 & \alpha & \beta \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & \alpha - 1 & \beta - 5 \end{array}\right]$$

$$\xrightarrow{R_3 - \frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & \alpha - 2 & \beta - 7 \end{array}\right]$$

Now, for infinite solution last row must be completely zero

i.e. $\alpha - 2$ and $\beta - 7 = 0$

$\Rightarrow$ $\alpha = 2$ and $\beta = 7$

T13 : Solution

(d)

$$x + y + z = 4 \quad \text{...(1)}$$

$$x - y + z = 0 \quad \text{...(2)}$$

$$2x + y + z = 5 \quad \text{...(3)}$$

Adding (1) and (2) & (2) and (3) gives

$2x + 2z = 4$ and $3x + 2z = 5$ which gives $x = 1$, $z = 1$ and $y = 2$

Alt: Option (b) can be eliminated since they do not satisfy 1st condition. Only (d) satisfies 3rd equation.

T14 : Solution

(b)

$$\text{Consider } (AB) = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & a & b \end{bmatrix}$$

$R_2 - 2R_1$, $R_3 - 5R_1$

$$= 2\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & a-10 & b-10 \end{bmatrix}$$

$R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & a-8 & b-6 \end{bmatrix}$$

$a = 8$

$b = 6$

$\therefore$ Infinite many solutions.

T15 : Solution

(c)

$(1 - \lambda)x + 2y + 3z = 0$

$3x + (1 - \lambda)y + 2z = 0$

$2x + 3y + (1 - \lambda)z = 0$

$\because$ System of equations has non-zero solution.

$\therefore \quad |A| = 0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 \\ 3 & (1-\lambda) & 2 \\ 2 & 3 & (1-\lambda) \end{vmatrix} = 0$$

$\Rightarrow \quad (1 - \lambda)[(1 - \lambda)^2 - 6] - 2[3 - 3\lambda - 4] + 3[9 - 2 + 2\lambda] = 0$

$\Rightarrow \quad (1 - \lambda)[\lambda^2 + 1 - 2\lambda - 6] - 2[-3\lambda - 1] + 3[7 + 2\lambda] = 0$

$\Rightarrow \quad -\lambda^3 + 3\lambda^2 + 15\lambda + 18 = 0$

$\Rightarrow \quad \lambda^3 - 3\lambda^2 - 15\lambda - 18 = 0$

$\Rightarrow \quad (\lambda - 6)(\lambda^2 + 3\lambda + 3) = 0$

$\because \quad \lambda - 6 = 0$ and $\lambda^2 + 3\lambda + 3 \neq 0$ $(D = 9 - 12 < 0)$

$\therefore$ $\lambda = 6$ is the only possible value for which system has a non-zero solution.

T16 : Solution

(d)

For non trivial solution,

$$|A| = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 3k-8+3+3 & 3 & 3 \\ 3+3k-8+3 & 3k-8 & 3 \\ 3+3+3k-8 & 3 & 3k-8 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3k-2 & 3 & 3 \\ 3k-2 & 3k-8 & 3 \\ 3k-2 & 3 & 3k-8 \end{vmatrix} = 0$$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3k-8 & 3 \\ 1 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_2 - R_1$, $R_3 - R_1$

$$(3k-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3k-11 & 0 \\ 0 & 0 & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)(3k-11)^2 = 0$$

$$k = \frac{2}{3}, \frac{11}{3}, \frac{11}{3}$$

T17 : Solution

(b)

Although λ_i^m will be the corresponding eigen values of A^m, x_i^m need not be corresponding eigen vectors.

T18 : Solution

(d)

Sum of eigen values = Tr(A) = –1 + –1 + 3 = 1

So, $\Sigma \lambda_i = 1$

Only choice (d) (3, –1 + 3j, –1 – 3j) gives $\Sigma \lambda_i = 1$.

T19 : Solution

(b)

$\because$ Sum of eigen value of the matrix

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} \ldots\ldots \frac{1}{n(n+1)}$$

$$= \left[\frac{1}{1}-\frac{1}{2}\right]+\left[\frac{1}{2}-\frac{1}{3}\right]+\left[\frac{1}{3}-\frac{1}{4}\right]\ldots\ldots\left[\frac{1}{n}-\frac{1}{n+1}\right]$$

$$= \left[1-\frac{1}{n+1}\right]$$

T20 : Solution

(c)

$\because$ For lower triangle matrix eigen value are the element of main diagonal.

$\therefore$ Product of eigen value of the matrix

$$= \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\ldots\ldots\frac{1}{n}$$

$$= \frac{1}{1\cdot 2\cdot 3\cdot 4\ldots\ldots n}$$

$$= \frac{1}{n!}$$

T21 : Solution

(b)

For a matrix containing complex number, eigen values are real if and only if

$$A = A^{\theta} = (\bar{A})^T$$

$$A = \begin{bmatrix} 10 & 5+j & 4 \\ x & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})^T = \begin{bmatrix} 10 & \bar{x} & 4 \\ 5-j & 20 & 2 \\ 4 & 2 & -10 \end{bmatrix}$$

By comparing these,

$$x = 5 - j$$

T22 : Solution

(a)

Eigen values of B is are

(i) $3^2 - 3 = 6$

(ii) $2^2 - 2 = 2$

(iii) $(-1)^2 + 1 = 2$

$\therefore$ $|B| = 6 \times 2 \times 2 = 24$

T23 : Solution

(d)

$$\text{Trace of } (I + M + M^2) = \text{tr}(I) + \text{tr}(M) + \text{tr}(M^2)$$

$$\text{Trace of } M = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$$

$$\text{Trace of } M^2 = 1^2 + (\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2 + (\alpha^4)^2$$

$$\text{Trace of } I = 1 + 1 + 1 + 1 + 1 = 5$$

$\because$ $\alpha = e^{2\pi i/5}$

α is the fifth root of unity or $x^5 = \alpha$.

$\Rightarrow$ $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

$$\begin{aligned}\text{tr}(M^2) &= 1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 \\ &= 1 + \alpha^2 + \alpha^4 + \alpha + \alpha^3 \\ &= 0\end{aligned}$$

$\Rightarrow$ Trace of $(I + M + M^2) = 5 + 0 + 0 = 5$

T24 : Solution

The constant term of the polynomial = Determinant of the matrix

= Product of eigen values

$\because$ $$\text{Matrix } I = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 3 & 2 & -5 \\ 2 & 4 & 12 & 10 \\ 3 & -2 & 1 & -4 \end{bmatrix}$$

$$|I| = \begin{bmatrix} 1 & 2 & 6 & 5 \\ -1 & 3 & 2 & -5 \\ 2 & 4 & 12 & 10 \\ 3 & -2 & 1 & -4 \end{bmatrix}$$

$$R_2 \to R_2 + R_1;\ R_3 \to R_3 - 2R_1$$

$$|I| = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 0 & 5 & 8 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

$$|I| = 0$$

Hence, constant term will be zero.

■■■■

2 Calculus

Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$y = x + \sqrt{x + \sqrt{x + \cdots}}$$

$$(y - x)^2 = x + \sqrt{x + \sqrt{x + \cdots}}$$

$$y^2 + x^2 - 2xy = y$$

$x = 2$

$$y^2 + 4 - 4y = y$$

$$y^2 - 5y + 4 = 0$$

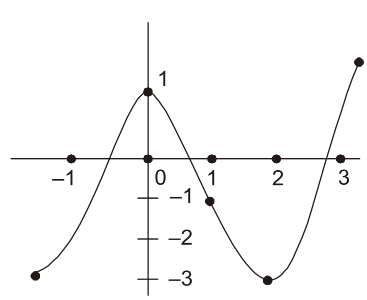
y = 1, 4

$$y(2) = 2 + \sqrt{2 + \sqrt{2 + \cdots}} > 2$$

$\therefore$ $y(2) = 4$ only

T2 : Solution

(b)



$$f(x) = x^3 - 3x^2 + 1$$

$$f'(x) = 3x^2 - 6x$$

$$f'(x) = 0$$

$$3x^2 - 6x = 0$$

$$3x(x-2) = 0$$
$$x = 0, 2$$
$$f''(x) = 6x - 6$$

At $x = 0$ $\quad f''(0) = -6$ maxima

$x = 2$ $\quad f''(2) = 6$ minima

T3 : Solution

$$|4x-7| = 5$$
$$4x - 7 = -5$$
$$4x = 2$$
$$x = \frac{1}{2}$$

$$2|x| - |-x|$$

$$2\left|\frac{1}{2}\right| - \left|-\frac{1}{2}\right|$$

$$2\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right) = \frac{1}{2}$$

$$4x - 7 = 5$$
$$4x = 12$$
$$x = 3$$

$$2|x| - |-x|$$

$$2|3| - |-3|$$

$$2(3) - 3 = 3$$

T4 : Solution

$$u_x = \begin{vmatrix} 2x & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Similarly $u_y = 0 \quad u_z = 0$

$\therefore \qquad u_x + u_y + u_z = 0$

T5 : Solution

(c)

$$y = |2-3x| = 2 - 3x \quad 2 - 3x \geq 0$$
$$= 3x - 2 \quad 2 - 3x < 0$$

Therefore, $\qquad y = 2 - 3x \quad x \leq \frac{2}{3}$

$$= 3x - 2 \quad x > \frac{2}{3}$$

Since $2 - 3x$ and $3x - 2$ are polynomials, these are continuous at all points. The only concern is at $x = 2/3$

Left limit at $x = \frac{2}{3}$ is $2 - 3 \times \frac{2}{3} = 0$.

Right limit at $x = \frac{2}{3}$ is $3 \times \frac{2}{3} - 2 = 0$.

$$f\left(\frac{2}{3}\right) = 2 - 3 \times \frac{2}{3} = 0$$

Since, Left limit = Right limit = $f\left(\frac{2}{3}\right)$,

Function is continuous at $\frac{2}{3}$.

y is therefore continuous $\forall\, x \in R$

Now since $2 - 3x$ and $3x - 2$ are polynomials, they are differentiable.

only concern is at $x = \frac{2}{3}$.

Now, at $x = \frac{2}{3}$, LD = Left derivative = – 3

$$\text{RD} = \text{Right derivative} = +3$$

$$\text{LD} \neq \text{RD}$$

$\therefore$ The function y is not differentiable at $x = \frac{2}{3}$

So, we can say that y is differentiable $\forall x \in R$, except at $x = \frac{2}{3}$.

T6 : Solution

(c)

If $f(x)$ is continuous at $x = \frac{\pi}{2}$

$$\lim_{x \to \frac{\pi}{2}} \frac{\lambda \cos x}{\frac{\pi}{2} - x} = f\left(\frac{\pi}{2}\right) = 1 \qquad \text{...(i)}$$

Since the limit is in form of $\frac{0}{0}$, we can use L' hospitals rule on LHS of equation (i) and get

$$\lim_{x \to \frac{\pi}{2}} \frac{-\lambda \sin x}{-1} = 1$$

$$\Rightarrow \qquad \lambda \sin\frac{\pi}{2} = 1$$

$$\Rightarrow \qquad \lambda = 1$$

T7 : Solution

(c)

$$|x| = x \qquad x \geq 0$$
$$= -x \qquad x < 0$$

at $x = 0$ left limit = 0

Right limit = −0 = 0

$f(0) = 0$

Since left limit = Right limit = $f(0)$

$$\boxed{\text{So } |x| \text{ is continuous at } x = 0}$$

Now, LD = Left derivative (at $x = 0$) = −1

RD = Right derivative (at $x = 0$) = +1

LD ≠ RD

$$\boxed{\text{So } |x| \text{ is not differentiable at } x = 0}$$

So $|x|$ is continuous and non-differentiable at $x = 0$

T8 : Solution

(d)

$f(x)$ is continuous at any point

if $$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

T9 : Solution

(b)

P : If $f(x)$ is continuous at $x = x_0$, then it is also differentiable at $x = x_0$

Q : If $f(x)$ is continuous at $x = x_0$, then it may or may not be derivable at $x = x_0$

R : If $f(x)$ is differentiable at $x = x_0$, then it is also continuous at $x = x_0$

P is false

Q is true

R is true Option (b) is correct

T10 : Solution

(a)

$$\begin{cases} 2, & \text{if } x = 3 \\ x - 1, & \text{if } x > 3 \\ \dfrac{x+3}{3}, & \text{if } x < 3 \end{cases}$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+3}{3} = 2$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x - 1 = 2$$

Also, $f(3) = 2$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

So it is continuous at $x = 3$

option (a) is correct.

T11 : Solution

At $x = 0$

$$\frac{\sin 0}{e^0 \cdot 0} = \frac{0}{0}$$

So applying L'Hospitals rule,

$$\lim_{x\to 0} \frac{\cos x}{xe^x + e^x} = \frac{1}{0+1} = 1$$

T12 : Solution

$$\lim_{x\to 0} \frac{x - \sin x}{1 - \cos x}$$

Applying L' Hospital's rule

$$\lim_{x\to 0} \frac{\frac{d}{dx}(x - \sin x)}{\frac{d}{dx}(1 - \cos x)} = \lim_{x\to 0} \frac{(1 - \cos x)}{\sin x}$$

(It is still of $\frac{0}{0}$ form)

Again applying L' Hospital's rule

$$\lim_{x\to 0} \frac{\frac{d}{dx}(1 - \cos x)}{\frac{d}{dx}(\sin x)} = \lim_{x\to 0} \frac{\sin x}{\cos x} = \frac{0}{1} = 0$$

T13 : Solution

$\lim_{x\to 0} \frac{(e^{2x} - 1)}{\sin 4x}$, it is of $\left(\frac{0}{0}\right)$ from

Applying L' Hospital's rule,

$$\lim_{x\to 0} \frac{2e^{2x}}{4\cos 4x} = \frac{2\times 1}{4\times 1} = \frac{1}{2}$$

T14 : Solution

$$\lim_{x\to 0}\left(\frac{-\sin x}{2\sin x + \cos x}\right) = \lim_{x\to 0}\left(\frac{-\sin 0}{2\sin 0 + \cos 0}\right) = \frac{0}{1} = 0$$

(**Note:** Since the function is not evaluating to 0/0 not need to use L' Hospital's rule)

T15 : Solution

(d)

(a) $f(x)$ = tan (πx) is not continuous at $x = \frac{1}{2}$.

(b) $f(x)$ is continuous at $x = \frac{1}{2}$ but not differentiable.

(c)
$$f(0) = 0$$
$$f(1) = (1)^2 = 1$$
and
$$f(0) \neq f(1)$$

(d)
$$f(x) = \sqrt{x(1-x)}$$

$f(x)$ is continuous in [0, 1] and differentiable in (0, 1) and $f(0) = f(1) = 1$.
So, all these conditions are satisfied. Hence, Rolle's theorem is valid.

T16 : Solution

(d)
Maclaurin series expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots$$

at $x = 1$,
$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \ldots\ldots$$

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

T17 : Solution

(1)
Maclaurin series : Expansion of sin x is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots$$

at $x = \frac{\pi}{2} \Rightarrow \lim_{x \to \frac{\pi}{2}} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots\ldots \right) = \sin\left(\frac{\pi}{2}\right) = 1$

T18 : Solution

(c)
We need absolute maximum of
$f(x) = x^3 - 9x^2 + 24x + 5$ in the interval [1, 6]
First find local maximum if any by putting $f'(x) = 0$.

i.e. $f'(x) = 3x^2 - 18x + 24 = 0$

i.e. $x^2 - 6x + 8 = 0$

$x = 2, 4$

Now $f''(x) = 6x - 18$

$f''(2) = 12 - 18 = -6 < 0$ (So $x = 2$ is a point of local maximum)

and $f''(4) = 24 - 18 = +6 > 0$ (So $x = 4$ is a point of local minimum)

Now tabulate the values of f at end point of interval and at local maximum point, to find absolute maximum in given range, as shown below:

x	$f(x)$
1	21
2	25
6	41

Clearly the absolute maxima is at $x = 6$
and absolute maximum value is 41.

T19 : Solution

(d)
$f(x)$ has a local minimum at $x = x_0$

if $f'(x_0) = 0$

and $f''(x_0) > 0$

T20 : Solution

$$f(x) = x^3 - 9x^2 + 24x + 5$$
$$f'(x) = 3x^2 + 18x + 24$$
$$f''(x) = 6x - 18$$
$$f'(x) = 0$$
$$3x^2 + 18x + 24 = 0$$
$$x^2 - 6x + 8 = 0$$
$$x = 2, 4$$

$x = 2,$ $f''(2) = -6$

$\therefore$ Relative maxima

$x = 4,$ $f''(4) = 6$ relative minima

$\therefore$ Maximum value of $f(x) = f(2) = 25$.

T21 : Solution

(6)

$$f(x) = 2x^3 - 9x^2 + 12x - 3$$
$$f'(x) = 6x^2 - 18x + 12$$
$$f''(x) = 12x - 18$$
$$f'(x) = 0$$
$$x^2 - 3x + 2 = 0$$
$$x = 1, x = 2$$
$$f''(1) = -6;\ f''(2) = 6$$

$x = 1$ is local maxima point.

$$\text{Global maxima of } f(x) = \text{Max}\{f(0), f(1), f(3)\}$$
$$= \{-3, 2, 6\} = 6$$

T22 : Solution

(10)

Given : $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3$

$$= 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3} + 3$$

$$= 5\cos\theta + \frac{3}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$$

$\because$ The maximum value of a sin θ + β cos θ is $\sqrt{a^2 + b^2}$.

$\therefore$ Maximum value of $\frac{13}{2}\cos\theta - \frac{3\sqrt{2}}{2}\sin\theta$ is

$$= \sqrt{\left(\frac{13}{2}\right)^2 + \left(\frac{-3\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{169}{4} + \frac{27}{4}}$$

$$= \sqrt{49} = 7$$

So maximum value of $\frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta + 3$ is

$$= 7 + 3 = 10$$

T23 : Solution

(d)

Given : $$f(x, y) = \frac{ax^2 + by^2}{xy} = a\left(\frac{x}{y}\right) + b\left(\frac{y}{x}\right)$$

$$\frac{\partial f}{\partial x} = \frac{a}{y} - \frac{by}{x^2}\ ; \ \frac{\partial f}{\partial y} = -\frac{ax}{y^2} + \frac{b}{x}$$

$$\left.\frac{a}{y} - \frac{by}{x^2}\right|_{\substack{x=1\\y=2}} = \left.-\frac{ax}{y^2} + \frac{b}{x}\right|_{\substack{x=1\\y=2}}$$

$$\frac{a}{2} - \frac{2b}{1} = -\frac{a}{4} + b$$

$$\frac{3a}{4} = 3b$$

$$a = 4b$$

T24 : Solution

(d)

$$x^a y^b = (x + y)^{a+b}$$

Differentiating w.r.t x

$$= ax^{a-1}y^b + bx^a y^{b-1}\frac{dy}{dx}$$

$$= (a+b)(x+y)^{(a+b-1)}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{ax^a y^b}{x} + \frac{bx^a y^b}{y} \times \frac{dy}{dx} = (a+b)\frac{(x+y)^{a+b}}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$ax^a y^b\left[\frac{a}{x} + \frac{by}{yx}\right] = (a+b)\frac{(x+y)^{a+b}}{x+y}\left(1 + \frac{dy}{dx}\right)$$

$$\frac{a}{x} - \frac{a+b}{x+y} = \left(\frac{a+b}{x+y} - \frac{b}{y}\right)\frac{dy}{dx}$$

$$\frac{ax + ay - ax - bx}{(x)(x+y)} = \left(\frac{ay + by - bx - by}{(y)(x+y)}\right)\frac{dy}{dx}$$

$$\frac{ay - bx}{x(x+y)} = \left(\frac{ay - bx}{y(x+y)}\right)\frac{dy}{dx}$$

$$= \frac{y}{x}$$

T25 : Solution

(c)

$$w = f(x,\ y)$$

By chain rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \times \frac{dx}{dt} + \frac{\partial w}{\partial y} \times \frac{dy}{dt}$$

T26 : Solution

(a)

$$Z = f(x,\ y)$$
$$x = e^{u} + e^{-v}$$
$$y = e^{-u} - e^{v}$$
$$z_u = \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$
$$= \frac{\partial z}{\partial x} \cdot e^{u} + \frac{\partial z}{\partial y} \cdot (-e^{-u})$$
$$= e^{u} z_x - e^{-u} z_y$$
$$z_v = \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$
$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$
$$= -e^{-v} z_x - e^{v} z_y$$
$$z_u - z_v = e^{u} \cdot z_x - e^{-v} z_y - (-e^{-v} z_x - e^{v} z_y)$$
$$= x z_x - y z_y$$

T27 : Solution

(c)

$$u = x^{n} f_1\left(\frac{x}{y}\right) + y^{-n} f_2\left(\frac{x}{y}\right)$$

Since u is homogeneous f degree n.

So
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \qquad \text{...(i)}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (n)(n-1)u \qquad \text{..(ii)}$$

Adding (i) and (ii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 y}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = nu + (n^2 - n)u = n^2\, u$$

T29 : Solution

(b)

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}0 = \frac{\pi}{2}$$

and $$L(\sin x) = \frac{1}{s^2+1}$$

$\Rightarrow$ $$L\left(\frac{\sin x}{x}\right) = \int_s^\infty \frac{1}{s^2+1}\,dx$$

(Using "division by x")

$$= \left[\tan^{-1}s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}(s) = \cot^{-1}(s)$$

$\Rightarrow$ $$\int_0^\infty e^{-sx}\frac{\sin x}{x}\,dx = \cot^{-1}(s)$$ (Using definition of Laplace transform)

Put $s = 0$,

we get $$\int_0^\infty \frac{\sin x}{x}\,dx = \cot^{-1}(0) = \frac{\pi}{2}$$

$$\frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}\,dx = \pi$$

T30 : Solution

(a)

Let, $$\sin^{-1}x = t$$

$$\frac{dx}{\sqrt{1-x^2}} = dt$$

$$I = \int_0^{\pi/2} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{\pi/2} = \frac{\pi^3}{24}$$

T31 : Solution

(b)

$$I = \int_0^{\pi/2} \log(\sin x)\,dx$$

$$= \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2}-x\right)\right)dx$$

$$I = \int_0^{\pi/2} \log\cos x\,dx$$

$$2I = \int_0^{\pi/2} \log\sin x.dx + \int_0^{\pi/2} \log\cos x\,dx$$

$$= \int_0^{\pi/2} \log \sin x . \cos x . dx$$

$$= \int_0^{\pi/2} \log 2 \sin x . \cos x . dx - \int_0^{\pi/2} \log 2 dx$$

$$= \int_0^{\pi/2} \log \sin 2x . dx - \int_0^{\pi/2} \log 2 dx$$

$$= \frac{1}{2}\int_0^{\pi} \log \sin u \, du - \frac{\pi}{2}\log 2$$

$$= \int_0^{\pi/2} \log \sin u \, du - \frac{\pi}{2}\log 2$$

$$= I - \frac{\pi}{2}\log 2$$

$$I = -\frac{\pi}{2}\log 2$$

T32 : Solution

(b)

$$I = \int_0^{\infty} e^{-x^2/2} dx \text{ or } I = \int_{-\infty}^{\infty} e^{-y^2/2} dx$$

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2/2} dx \int_{-\infty}^{\infty} e^{-y^2/2} dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)/2} dx \, dy$$

$$r = \sqrt{x^2+y^2}$$

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-r^2/2} dr$$

$$= 2\pi \int_0^{\infty} e^{-t} dt \qquad \left[\text{Let } \frac{r^2}{2} = t, \, rdr = dt\right]$$

$$= 2\pi \left.\frac{e^{-t}}{-1}\right|_0^{\infty} = 2\pi$$

$$I^2 = 2\pi$$

$$\Rightarrow \qquad I = \sqrt{2\pi}$$

T33 : Solution

(b)

$$I = \int_0^n \frac{1-\left(\frac{x}{n}\right)^n}{n-x}dx = \int_0^n \frac{1-\left(\frac{x}{n}\right)^n}{1-\left(\frac{x}{n}\right)}\frac{dx}{n}$$

Let, $\frac{x}{n} = t \quad x \to 0 \quad t \to 0$

$x \to n \quad t \to 1$

$$\frac{dx}{n} = dt$$

$$\Rightarrow \quad I = \int_0^1 \frac{1-t^n}{1-t}dt$$

$$\frac{1-t^n}{1-t} = \sum_{r=1}^{n} t^{r-1}$$

$$I = \int_0^1 \sum_{r=1}^{n} t^{r-1}dt$$

Changing order of summation and integral

$$I = \sum_{r=1}^{n}\int_0^1 t^{r-1}dt = \sum_{r=1}^{n}\left.\frac{t^r}{r}\right|_0^1 = \sum_{r=1}^{n}\frac{1}{r}$$

$$= 1+\frac{1}{2}+\frac{1}{3}\cdots\frac{1}{n}$$

T34 : Solution

$$I = \int_0^{\pi/2}\frac{\cos x-\sin x}{1+\sin x\cos x}dx$$

$$I = \int_0^{\pi/2}\frac{\cos\left(\frac{\pi}{2}-x\right)-\sin\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\pi/2}\frac{\sin x-\cos x}{1+\sin x\cos x}dx$$

$$= -\int_0^{\pi/2}\frac{\cos x-\sin x}{1+\sin x\cos x}dx$$

$$I = -I$$

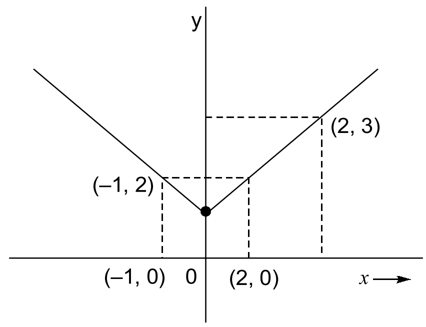
$$2I = 0$$

$$\Rightarrow \quad I = 0$$

T36 : Solution

$$\int_{-1}^{2}(1+|x|)\,dx$$

$y = 1 + |x|$



$$\int_{-1}^{2}(1+|x|)\,dx = \text{Area under the curve}$$

$$= \int_{-1}^{2} y.dx = \frac{1}{2}[(1+2)\times 1+(1+3)\times 2]$$

$$= 5.5$$

T37 : Solution

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \left(\sin^{-1}x\right)\Big|_0^1 = \sin^{-1}1 = \frac{\pi}{2}$$

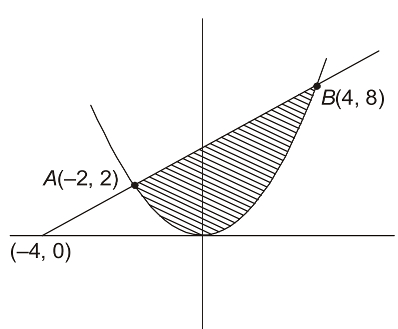
T38 : Solution

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}dx = 2\int_0^{\infty}\frac{1}{1+x^2}dx$$

$$= 2\left(\tan^{-1}x\right)\Big|_0^{\infty} = 2\tan^{-1}\infty = \pi$$

T39 : Solution

(b)



Point of intersection

$$x = \frac{x^2}{2} - 4$$

$$2x = x^2 - 8$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$x = -2, x = 4$$

Point of intersection are (–2, 2), (4, 8)

$$\text{Area} = \int_{-2}^{4} \int_{y=\frac{x^2}{2}}^{y=x+4} dydx$$

$$= \int_{-2}^{4} \left[x + 4 - \frac{x^2}{2}\right] dx$$

$$= \frac{x^2}{2} + 4x - \frac{x^3}{6}\Big|_{-2}^{4}$$

$$= \frac{1}{2}(16-4) + 4(4+2) - \frac{1}{6}(64+8)$$

$$= 6 + 24 - 12 = 18$$

T40 : Solution

$$\int_1^2 \left[\int_0^x \frac{1}{x^2+y^2} dy\right] dx$$

$$= \int_1^2 \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]\Big|_0^x dx$$

$$= \int_1^2 \frac{\pi}{4} \cdot \frac{1}{x} dx = \frac{\pi}{4}(\log x)\big|_1^2 = \frac{\pi}{4}\log 2$$

T41 : Solution

$$\int_0^\infty \left[\int_0^y y \cdot e^{-y} \cdot e^{-x} dx\right] dy$$

$$\int_0^\infty ye^{y}\left(\frac{e^{-x}}{-1}\right)\Big|_0^y dy = \int_0^\infty ye^{-y}\left(\frac{e^{-y}-1}{-1}\right)dy$$

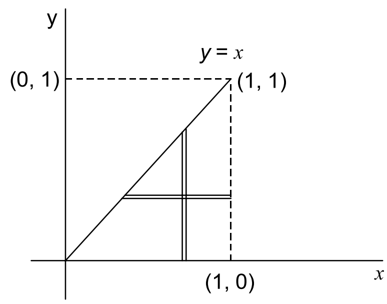
$$= \int_0^\infty ye^{-y}(1-e^{-y})dy = \frac{3}{4}$$

T42 : Solution

(d)

$$I = \int_0^1 \int_y^1 y\sqrt{1+x^3}\, dx\, dy$$

$$I = \int_0^1 \int_0^x y\sqrt{1+x^3}\, dx\, dy$$



$$I = \int_0^1 \left.\frac{y^2}{2}\right|_0^x \sqrt{1+x^3}\, dx = \frac{1}{2}\int_0^1 x^2\sqrt{1+x^3}\, dx$$

Let $1 + x^3 = t$

$$3x^2 dx = dt$$

$$= \frac{1}{2} \times \frac{1}{3}\int_1^2 \sqrt{t}\, dt$$

$$= \left.\frac{1}{6}\frac{t^{3/2}}{3/2}\right|_1^2$$

$$= \frac{1}{9}\left[2^{3/2} - 1^{/32}\right] = \frac{1}{9}\left(2\sqrt{2} - 1\right)$$

T43 : Solution

(a)

$$\phi(r, \theta) = J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} x_r & y_r \\ x_\theta & y_\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r$$

■■■■

3 Vector Calculus

Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$f = x^2 + 3y^2 + 2z^2$$

$$\Delta f = \text{grad } f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(2x) + j(6y) + k(4z)$$

The gradient at $P(1, 2, -1)$ is

$$= i(2 \times 1) + j(6 \times 2) + k(4 \times -1)$$

$$= 2i + 12j - 4k$$

T2 : Solution

(a)

$$\phi = xy^2 + yz^2 + zx^2$$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}\left(y^2 + 2xz\right) + \bar{j}\left(2xy + z^2\right) + \bar{k}\left(2yz + x^2\right)$$

$$\nabla\phi_{(2,-1,1)} = \bar{i}(1+4) + \bar{j}(-4+1) + \bar{k}(-2+4)$$

$$= 5\bar{i} - 3\bar{j} + 2\bar{k}$$

$$\bar{P} = \bar{i} + 2\bar{j} + 2\bar{k}$$

$$|\bar{P}| = \sqrt{1+4+4} = 3$$

The directional derivative of $\phi(x, y, z)$ at $(2, -1, 1)$ in the direction of $\bar{P}$ is $\nabla\phi_{\text{at } P} \cdot \dfrac{\bar{P}}{|\bar{P}|}$

$$= \left(5\bar{i} - 3\bar{j} + 2\bar{k}\right) \cdot \left(\frac{\bar{i} + 2\bar{j} + 2\bar{k}}{3}\right)$$

$$= \frac{5 - 6 + 4}{3} = 1$$

T3 : Solution

(d)

$$\text{div}\{(x-y)\hat{i} + (y-x)\hat{j} + (x+y+z)\hat{k}\} = \frac{\partial}{\partial x}(x-y) + \frac{\partial}{\partial y}(y-x) + \frac{\partial}{\partial z}(x+y+z) = 3$$

T4 : Solution

(d)

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \vec{A} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$$\nabla \cdot \vec{A} = 3$$

T5 : Solution

(a)

$$\vec{F} = x^2 z^2 \vec{i} - 2xy^2 z\vec{j} + 2y^2 z^3 \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 z^2 & -2xy^2 z^2 & 2y^2 z^3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(2y^2 z^3) + \frac{\partial}{\partial z}(2xy^2 z)\right]$$

$$- \vec{j}\left[\frac{\partial}{\partial y}(2y^2 z^3) - \frac{\partial}{\partial z}(x^2 z^2)\right]$$

$$+ \vec{k}\left[\frac{\partial}{\partial x}(-2xy^2 z) - \frac{\partial}{\partial y}(x^2 z^2)\right]$$

$$\nabla \times \vec{F} = \vec{i}[4yz^3 + 2xy^2] - \vec{j}[2zx^2]$$

$$+ \vec{k}[-2y^2 z - 0]$$

$$= (4yz^3 + 2xy^2)\vec{i} - (2x^2 z)\vec{j} - (2y^2 z)\vec{k}$$

T6 : Solution

$A(0, 2, 1)$ and $B(4, 1, -1)$

The equation of the line AB is

$$\frac{x-0}{4-0} = \frac{y-2}{1-2} = \frac{3-1}{-1-1} = t \quad \text{say}$$

$x = 4t$; $y = -t + 2$; $z = -2t + 1$

$dx = 4dt$; $dy = -dt$; $dz = -2dt$

t varies from 0 to 1

$$I = \int_0^1 2(-2t+1)\,4dt + 2(-t+2)\,(-dt) + 2(4t)\,(-2dt)$$

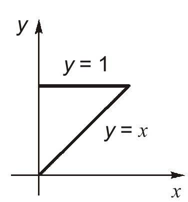
$$= \int_0^1 (-16t + 8 + 2t - 4 - 16t)dt$$

$$= \int_0^1 (-30t + 4)dt$$

$$= \left(-30\frac{t^2}{2} + 4t\right)\Bigg|_0^1 = -15 + 4 = -11$$

T7 : Solution

(b)



$$F_1 = x^2 + 2\sin^2 x \cos x \qquad F_2 = 4x + y^2$$

$$\frac{\partial F_1}{\partial y} = 0 \qquad \frac{\partial F_2}{\partial x} = 4$$

$$\int_c F_1\,dx + F_2\,dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

$$= \iint (4-0)\,dx\,dy = 4 \cdot \left(\frac{1}{2}\right) = 2$$

T8 : Solution

$$M = xy^2 + 2y + \sin e^x$$

$$\frac{\partial M}{\partial y} = 2xy + 2$$

$$N = x^2 y + \cos e^y$$

$$\frac{\partial M}{\partial y} = 2xy$$

From Green's theorem,

$$I = \int_{-1}^{1}\int_{-1}^{1}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$= \int_{-1}^{1}\int_{-1}^{1}(2xy - 2xy - 2)dxdy$$

$$= -2\int_{-1}^{1}\int_{-1}^{1}dxdy$$

$$= -2 \times (1 + 1) \times (1 + 1)$$

$$= -8$$

T9 : Solution

$$I = \int_c e^x dx + 2ydy - dz$$

Stoke's Theorem

$$\int_c F.dr = \iint_R (\nabla \times F)\cdot \hat{n}ds$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$= 0\hat{i} - 0\hat{j} + 0\hat{k}$$

Hence, $I = 0$.

T10 : Solution

(a)

$$\bar{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \cdot \bar{F} = 3$$

By Gauss divergence theorem

$$\int_s \bar{F}\cdot \hat{n}ds = \int_v \nabla \cdot \bar{F}dv = \int_v 3dv = 3\,V$$

where V is volume of $x^2 + y^2 + z^2 = 4$

$$= 3\left(\frac{32\pi}{3}\right) = 32\,\pi$$

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4 Differential Equations

Detailed Explanation of Try Yourself Questions

T1 : Solution

$$(D^4 - D^2)y = 0$$

Auxiliary equation is

$$m^4 - m^2 = 0$$

$$m = 0, 0 \quad m = \pm 1$$

Complimentary function is

$$(C_1 + C_2x) + C_3e^{-x} + C_4 e^x = C_1 + C_2x + C_3e^{-x} + C_4 e^x$$

$$\text{Set of independent} = \{1, x, e^{-x}, e^x\}$$

T2 : Solution

(b)

$$\frac{dy}{dx} = (4x + y + 1)^2$$

Let, $4x + y + 1 = t$

Differentiating w.r.t. x

$$4 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dt}{dx} - 4$$

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\int \frac{dt}{t^2 + 4} = \int dx$$

$$\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c$$

T3 : Solution

(a)

Let the number of bacteria at time $t = n$

$$\frac{dn}{dt} = n$$

$$\frac{dn}{dt} = kn$$

$$\int_{n_o}^{2n_o}\frac{dn}{n} = \int_0^2 kdt$$

$$\log n\Big|_{n_o}^{2n_o} = kT\Big|_0^2$$

$$2k = \log 2$$

$$k = \frac{1}{2}\log 2$$

$$\Rightarrow \qquad \log\left(\frac{n}{n_0}\right) = \frac{1}{2}\log 2(t-0)$$

$$\log\left(\frac{3n_0}{n_0}\right) = \frac{1}{2}\log 2(t-0)$$

$$\log 3 = \frac{1}{2}\log 2t$$

$$\Rightarrow \qquad t = 2\frac{\log 3}{\log 2}$$

T4 : Solution

(a)

Let the rate of cooling is $\frac{dT}{dt}$ where T is temperature.

$$\frac{dT}{dt} \propto (T-T_0)$$

where, T is temperature of the body and T_o is temperature of the air.

$$\int_{100^\circ}^{75^\circ}\frac{dT}{T-T_o} = \int_0^1 -kt$$

$$In(T-T_0)\Big|_{100^\circ}^{75^\circ} = -kt\Big|_0^1$$

$$In\left(\frac{75^\circ-25^\circ}{100^\circ-25^\circ}\right) = -kt\Big|_0^1$$

$$ln\left(\frac{50°}{75°}\right) = -k$$

$$\Rightarrow \qquad k = ln\left(\frac{3}{2}\right)$$

After 3 minutes, temeperature = T

$$ln(T - T_0)\Big|_{100°}^{T} = -ln\left(\frac{3}{2}\right)\cdot t\Big|_0^3$$

$$ln\left(\frac{T-25°}{100° - 25°}\right) = -ln\frac{3}{2}\times 3$$

From here, $\qquad T = 47.22°C$

T5 : Solution

(d)

Let the amount of radium at time t is x grams.

$$\frac{dx}{dt} \propto -x$$

$$\frac{dx}{dt} = kx$$

$$x = x_o\, e^{-kt}$$

$\Rightarrow$ Let initial amount was 100%. 5% disappeared in 50 years

$\Rightarrow$ Remaining amount will be 95%

$$95 = 100\, e^{-k\cdot 50}$$

$$e^{-k\times 50} = \frac{95}{100}$$

After 100 years

$$x = 100 \times e^{-k\cdot 100}$$

$$x = 100\times\left(\frac{95}{100}\right)^2$$

$$= \frac{9025}{100} = 90.25\%$$

T6 : Solution

Differential equation $Mdx + Ndy = 0$ is exact.

If $\qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\because \qquad M = 27x^2 + Ky\cos x$

$$N = 2\sin x - 27y^3$$

$$\frac{\partial N}{\partial x} = 2\cos x$$

$$\frac{\partial M}{\partial y} = K\cos x$$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$2\cos x = K\cos x$$

Hence,
$$K = 2$$

T7 : Solution

(b)

$$\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$$

$$\frac{dy}{dx} + \frac{\sin x \sin y}{\cos x \cos y} = \cos x$$

$$\cos y \frac{dy}{dx} + \frac{\sin x \sin y}{\cos x} = \cos x$$

Let,
$$\sin y = t$$

$$\cos y \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + \delta \tan x \cdot t = \cos x$$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{ln \sec x}$$

$$= \sec x$$

$$y(\sec x) = \cos x \times \frac{1}{\sec x} dx$$

$$\sin y \cdot \sec x = x + c$$

$$\sin y = (x + c)\cos x$$

T8 : Solution

(d)

The equation can be re-written as

$$\frac{dy}{dx}\cos(x+y) + \frac{\sin(x+y)}{x} - e^x + \cos(x+y) = 0$$

$$\left[\frac{\sin(x+y)}{x} - e^x + \cos(x+y)\right]dx + \cos(x+y)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\cos(x+y)}{x} - \sin(x+y)$$

$$\frac{\partial N}{\partial x} = -\sin(x+y)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{\frac{\cos(x+y)}{x} - \sin(x+y) - (-\sin(x+y))}{\cos(x+y)}$$

$$= \frac{1}{x} \quad \text{[exclusive function of } x\text{]}$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx = e^{\ln x}}$$
$$= x$$

$\Rightarrow$ Differential equation becomes

$$\left[\sin(x+y) - xe^x + x\cos(x+y)\right]dx + x\cos(x+y)dy = 0$$

$$\int(\text{terms contain } 'x' \text{ only})dx + \int x\cos(x+y)dy = 0$$

$$\int -xe^x dx + \int x\cos(x+y)dy = 0$$

$$\int x\cos(x+y)dy = \int xe^x dx$$

$$x\sin(x+y) = xe^x - e^x + c$$

$$x\left(\sin(x+y) - e^x\right) + e^x = \text{Constant.}$$

T9 : Solution

The differential equation

$$(t^2-81)\frac{dy}{dt} + 5ty = \sin t$$

$$\frac{dy}{dt} + \left(\frac{5t}{t^2-81}\right)y = \frac{\sin t}{t^2-81}$$

$$P = \frac{5t}{t^2-81}$$

$$Q = \frac{\sin t}{t^2-81}$$

$$I.F = e^{\int P dt} = e^{\int \frac{5t}{t^2-81}dt}$$

$$= e^{5/2\int \frac{2t}{t^2-81}dt}$$

$$= e^{5/2 \ln(t^2-81)}$$

$$= (t^2-81)^{5/2}$$

Solution is
$$y(t^2-81)^{5/2} = \int \frac{\sin t}{(t^2-81)}\cdot(t^2-81)^{5/2}dt$$

$$y = \int \frac{\sin t}{(t^2-81)}dt + \frac{C}{(t^2-81)^{5/2}}$$

The solution exists for $t \neq \pm 9$. Hence only (a) option is not involving ± 9.

T10 : Solution

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}\log 2$$

$$= \frac{1}{D^2 - 2D + 2}(\log 2)\cdot e^{0x}$$

Put $D = 0$

$$= \frac{\log 2}{2}$$

T11 : Solution

$$\text{P.I.} = \frac{1}{D^3 - D^2 - D}(-e^x)$$

Put $D = 1$

$$= \frac{1}{1-1-1}(-e^x) = e^x$$

T12 : Solution

(d)

$$x^2\frac{dy}{dx} + 2xy = \frac{2\log x}{x}$$

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{2\log x}{x^3}$$

$$\text{I.F.} = e^{\int \frac{2}{x}dx} = 2^{2\ln x} = x^2$$

Solution of differential equation is

$$y\cdot \text{I.F.} = \int Q(x)\cdot \text{I.F.}\, dx$$

$$y\cdot x^2 = \int \frac{2\log x}{x^3}\cdot x^2 dx$$

$$y\cdot x^2 = 2\int \frac{\log x}{x}dx \qquad \left[\text{Let } \log x = t, \frac{1}{x}dx = dt\right]$$

$$y\cdot x^2 = 2\int t\,dt$$

$$y\cdot x^2 = 2\frac{t^2}{2} + C$$

$$y\cdot x^2 = (\log x)^2 + C$$

$$y(1) = 0$$

$$0\times 1 = \left[\log(1)^2\right] + C \Rightarrow C = 0$$

$$y(e) = ?$$

$$y\cdot e^2 = [\log e]^2 = 1$$

$$y = \frac{1}{e^2}$$

T13 : Solution

$$(x^3D^3 + 2x^2D^2 + 2)y = 0$$

Given equation is Cauchey-Euler equation.

$$x^3D^3 = \theta(\theta - 1)(\theta - 2)$$

$$x^2D^2 = \theta(\theta - 1) \text{ where } \theta = \frac{d}{dz} \text{ and } z = \ln(x)$$

$$[\theta(\theta - 1)(\theta - 2) + 2\theta(\theta - 1) + 2]y = 0$$

$$(\theta^3 - \theta^2 + 2)y = 0$$

AE is $\quad m^3 - m^2 + 2 = 0$

$$(m + 1)(m^2 - 2m + 2) = 0$$

$$m = -1,\ m = 1 \pm i$$

$$\text{C.F.} = C_1e^{-z} + e^z(C_2 \cos z + C_3 \sin z)$$

Solution is $\quad y = C_1x^{-1} + x(C_2 \cos(\ln x) + C_3 \sin(\ln x)).$

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5 Complex Variables

Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$\frac{(2-3i)}{(-5+i)} = \frac{(2-3i)}{(-5+i)} \times \frac{(-5-i)}{(-5-i)}$$

$$= \frac{-10-2i+15i-3}{25+1} = \frac{-13+13i}{26}$$

$$= -0.5 + 0.5i$$

T2 : Solution

(a)

Given : $\alpha \neq 1 \text{ and } \alpha^5 = 1$

$\because$ α is fifth root of unity.

Hence, $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

T3 : Solution

(b)

$$\frac{(2+i)^2}{(3-i)^2} = \frac{4-1+4i}{9-1-6i} = \frac{3+4i}{8-6i}$$

$$= \frac{3+4i}{2(4-3i)} \times \frac{4+3i}{4+3i}$$

$$= \frac{12-12+25i}{2(16+9)} = \frac{i}{2} = \frac{1}{2}\left[e^{i\pi/2}\right]$$

T4 : Solution

(c)

$$2^n\left[\frac{\sqrt{3}}{2}+\frac{i}{2}\right]^n+2^n\left[\frac{\sqrt{3}}{2}-\frac{i}{2}\right]^n$$

$$=2^n\left[\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right]^n+2^n\left[\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right]^n$$

$$=2^n e^{in\pi/6}+2^n e^{-in\pi/6}$$

$$=2^n\left[2\cos\frac{n\pi}{6}\right]=2^{n+1}\cos\frac{n\pi}{6}$$

T5 : Solution

$$f(z_1) = \frac{az_1+b}{cz_1+d} \text{ and } f(z_2) = \frac{az_2+b}{cz_2+d}$$

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

$$acz_1z_2+bcz_2+adz_1+bd=acz_1z_2+bcz_1+adz_2+bd$$

$$bc(z_2-z_1) = ad(z_2-z_1)$$

$$z_2 \neq z_1$$

$\Rightarrow$ $$bc = ad$$

$$d=\frac{bc}{a}=\frac{4\times 5}{2}=10$$

T6 : Solution

(d)

$$f = u+iv$$

$$u = 3x^2-3y^2$$

for f to be analysis, we have Cauchy-Riemann conditions,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{...}(i)$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \quad \text{...}(ii)$$

From (i) we have

$$6x = \frac{\partial v}{\partial y}$$

$\Rightarrow$ $$\int\partial v = \int 6x\partial y$$

$$v = 6xy+f(x)$$

i.e. $$v = 6xy+f(x) \quad \text{...}(iii)$$

Now applying equation (*ii*) we get

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\Rightarrow \quad -6y = -\left[6x + \frac{df}{dx}\right]$$

$$\Rightarrow \quad 6x + \frac{df}{dx} = 6y$$

$$\frac{df}{dx} = 6y - 6x$$

By integrating,

$$f(x) = 6yx - 3x^2 + K$$

Substitute in equation (*iii*)

$$v = 3x^2 + 6yx - 3x^2 + K$$

$$\Rightarrow \quad v = 6yx + K$$

T7 : Solution

(c)

As per Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 2y$$

and $$\frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial v}{\partial y} = 2y$$

$$\Rightarrow \quad v = y^2 + f(x)$$

$$\frac{\partial v}{\partial x} = 0 + f'(x) = -2x$$

$$\therefore \quad f(x) = -x^2 + \text{constant}$$

$$\therefore \quad v = y^2 - x^2 + \text{constant}$$

T8 : Solution

$$f(z) = \frac{\sin(z-1)}{z-1}$$

Maclaurin expansion of $$\sin(z-1) = (z-1) - \frac{(z-1)^3}{3!} + \frac{(z-1)^5}{5!} \ldots\ldots$$

$$f(z) = \left[1 - \frac{(z-1)^2}{3!} + \frac{(z-1)^4}{5!} \ldots\ldots\right]$$

$\because$ There is no terms in the principal root of $f(z)$. Hence function has removable singularity.

T9 : Solution

(c)

$$x(z) = \frac{1-2z}{z(z-1)(z-2)}$$

poles are $z = 0,\ z = 1$ and $z = 2$

Residue at $z = 0$

$$\text{residue} = \text{value of } \frac{1-2z}{(z-1)(z-2)} \text{ at } z = 0$$

$$= \frac{1-2\times 0}{(0-1)(0-2)} = \frac{1}{2}$$

Residue at $z = 1$

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-2)} \text{ at } z = 1$$

$$= \frac{1-2\times 1}{1(1-2)} = 1$$

Residue at $z = 2$

$$\text{residue} = \text{value of } \frac{1-2z}{z(z-1)} \text{ at } z = 2$$

$$= \frac{1-2\times 2}{2(2-1)} = -\frac{3}{2}$$

$\therefore$ The residues at its poles are $\frac{1}{2}, 1$ and $-\frac{3}{2}$.

T10 : Solution

$$f(z) = \frac{2z}{(z-1)^2(z-2)}$$

$z = 1$ is pole of order 2

$z = 2$ pole of order 1

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}\left((z-1)^2 \cdot \frac{2z}{(z-1)^2(z-2)}\right)$$

$$= \lim_{z\to 1}\left(\frac{(z-2)(2)-2z(1)}{(z-2)^2}\right)$$

$$= \frac{-2-2}{(1-2)^2} = -4$$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} (z-2)\cdot\frac{2z}{(z-1)^2(z-2)}$$

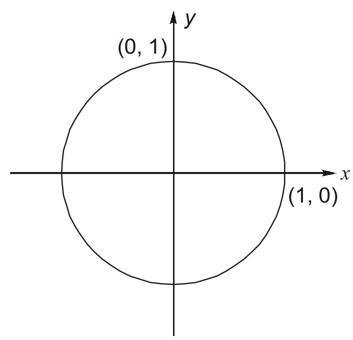
$$= \frac{4}{(2-1)^2} = 4$$

Sum of residues $= -4 + 4 = 0$

T11 : Solution

(b)

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues of } f(z) \text{ at all its pole lies inside } C)$$



$z = 0$ is the only pole inside C.

$$\underset{z=0}{\text{Res}}\ f(z) = \lim_{z\to 0}(z-0)\frac{\cos z}{z(z-2)(z-4)}$$

$$= \frac{1}{(-2)(-4)} = \frac{1}{8}$$

$$\int_{|z|=1} \frac{\cos z dz}{z(z-2)(z-4)} = 2\pi i \times \frac{1}{8}$$

$$= \frac{\pi i}{4}$$

T12 : Solution

(a)

$$I = \int_C \frac{e^{2z}}{(z+1)^4}dz \text{ where } C \Rightarrow |z| = 3$$

$z = -1$ is the point inside C.

$\because$ $$\int \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^n(a)$$

$$f(z) = e^{2z}$$

$$f'''(z) = 8e^{2z}$$

$$f'''(-1) = 8e^{-2}$$

$$\therefore \quad \int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{2\pi i}{3!} \times 8e^{-2}$$

$$= \frac{16\pi i}{6} \times e^{-2}$$

$$= \frac{8\pi i}{3} \times e^{-2}$$

T13 : Solution

(b)

(*i*) $Z_0 = 2$ -lies inside C,

So
$$\text{Res } f(z) = \lim_{z\to 2}(z-2)\cdot\frac{e^z}{z-2}$$
$$= e^2 = 7.39$$

$$\frac{1}{2\pi i}\int_c \frac{e^z}{z-2}dz = 2\pi i \cdot \frac{1}{2\pi i}(7.39) = 7.39$$

(*ii*) $Z_0 = -2$ lies out side C then
$$\text{Res } f(z) = 0$$

So
$$\int_c \frac{e^z}{z-2}dz = 2\pi i \frac{1}{2\pi i}(0) = 0$$

T14 : Solution

(b)

Singularities, $z = \frac{1}{2}, 2 \pm i$

only, $z = \frac{1}{2}$ lies inside C

By residue theorem,

$$\oint_C = 2\pi i(R) = \frac{48\pi i}{13}$$

Residue at $\frac{1}{2} = R_{1/2}$

$$= \lim_{z\to 1/2}\left[\left(z-\frac{1}{2}\right)\cdot\frac{2z+5}{\left(z-\frac{1}{2}\right)(z^2+4z+5)}\right] = \frac{24}{13}$$

Then the value of integral $= 2\pi i \times \frac{24}{13} = \frac{48\pi i}{13}$

T15 : Solution

(b)

From the diagram C is $y = x$

$$I = \int_C (x^2 + iy^2)\, dz$$

$$= \int_C (x^2 + iy^2)(dx + idy)$$

$$= \int_C (x^2 + ix^2)(dx + idx)$$

$$= \int x^2 dx + ix^2 dx + ix^2 dx - x^2 dx$$

$$= 2i\int_0^1 x^2 dx = 2i\left(\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{2i}{3}$$

T16 : Solution

$$I = \oint_c \frac{-3z+4}{(z^2+4z+5)}\, dz$$

$$= 2\pi i \text{ (sum of residues)}$$

Poles of $\dfrac{-3z+4}{(z^2+4z+5)}$ are given by

$$z^2 + 4z + 5 = 0$$

$$z = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Since the poles lie outside the circle $|z| = 1$.

So $f(z)$ is analytic inside the circle $|z| = 1$.

Hence $\oint_c f(z)dz = 2\pi i\,(0) = 0$

T17 : Solution

(a)

Given : $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2} dx$

$$I = \int_{-\infty}^{\infty} \frac{\sin z}{z^2+2z+2} dz$$

$$\sin z = \text{imaginary part of } e^{iz}$$

$$= \int_{-\infty}^{\infty} \frac{\text{I.P of } e^{iz}}{z^2+2z+2} dz$$

Poles are $z^2 + 2z + 2 = 0$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$z = -1 - i$$
$$\downarrow$$
Outside upper half
$$\downarrow$$
Residue is 0

$$-1 + i$$
$$\downarrow$$
inside upperhalf

$$\underset{z=-1+i}{\text{Res}}\ \phi(z)$$

$$= \lim_{z \to -1+i} z - (-1+i) \frac{e^{iz}}{(z-(-1+i))(z-(-1-i))}$$

$$= \frac{e^{i(-1+i)}}{(-1+i)-(-1-i)} = \frac{e^{-i-1}}{-1+i+1+i} = \frac{e^{-i-1}}{2i}$$

$$I = \text{I.P. of } 2\pi i \left(\frac{e^{-i-1}}{2i}\right) = \text{I.P of } \pi\left(e^{-i} \cdot e^{-1}\right)$$

$$= \text{I.P of } \pi e^{-1}(\cos 1 - i \sin 1) = \frac{-\pi \sin 1}{e}$$

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6 Probability and Statistics

Detailed Explanation *of* Try Yourself Questions

T1 : Solution

$$D = \{x_1, x_2, x_3, x_4\}$$

Total number of vector possible

$$= 2 \times 2 \times 2 \times 2 = 16$$

Combinations that product of first and third coordinate is '0' is

$$\begin{array}{|c|c|c|c|}\hline 0 & x_2 & 1 & x_3 \\ \hline\end{array} = 2 \times 2$$

$$\begin{array}{|c|c|c|c|}\hline 1 & x_2 & 0 & x_3 \\ \hline\end{array} = 2 \times 2$$

$$\begin{array}{|c|c|c|c|}\hline 0 & x_2 & 0 & x_3 \\ \hline\end{array} = 2 \times 2$$

Probability that product of first and third coordinate is '0' is

$$= \frac{4+4+4}{16} = \frac{12}{16} = \frac{3}{4}$$

T2 : Solution

Let $P(A)$ is probability of getting plumbring contract and $P(B)$ is probability of getting electrical contract.

$$P(A) = 0.5 \quad P(B) = 0.3$$

$$\overline{P(A \cup B)} = 0.25$$

$$P(A \cup B) = 1 - \overline{P(A \cup B)}$$

$$= 1 - 0.25 = 0.75$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = 0.5 + 0.3 - 0.75 = 0.05$$

T3 : Solution

	X	Y
Supply	60%	40%
Probability of supply	$\frac{60}{100}=\frac{3}{5}$	$\frac{40}{100}=\frac{2}{5}$
Reliable	96%	72%
Reliability Probability	$\frac{96}{100}$	$\frac{72}{100}$

By Baye's theorem,

Probability that shock absorber is reliable and from by

$$Y = \frac{\frac{72}{100}\times\frac{2}{5}}{\frac{72}{100}\times\frac{2}{5}+\frac{96}{100}\times\frac{3}{5}} = \frac{1}{3}$$

$$= 0.0334$$

T4 : Solution

(a)

$\boxed{4B \quad 3W}$ $\boxed{B \quad W \quad B \quad W \quad B \quad W \quad B}$ is the only one arrangement is favourable arrangement out of

$\frac{7!}{3!\,4!}$ arrangements.

$$\therefore \qquad \text{Required prob.} = \frac{1}{\frac{7\times6\times5\times4!}{3!\,4!}} = \frac{1}{35}$$

T5 : Solution

$\boxed{5\text{ Defective} \quad 15\text{ Non-defective}}$

$$\text{Required probability} = \frac{{}^{15}C_1 \cdot {}^{14}C_1}{{}^{20}C_1 \cdot {}^{19}C_1} = \frac{210}{380} = \frac{21}{38}$$

T6 : Solution

$$\begin{aligned}
\Sigma P(x) &= 1\\
10\,K^2 + 9\,K &= 1\\
10\,K^2 + 9\,K - 1 &= 0\\
10\,K^2 + 10\,K - K - 1 &= 0\\
10K\,(K+1) - (K+1) &= 0\\
(K+1)\,(10K-1) &= 0
\end{aligned}$$

$$K = -1 \qquad K = \frac{1}{10}$$

Probability any event is non-negative.

$\therefore$ $$K = 0.10$$

T7 : Solution

Sum of square of first 23 natural numbers

$$= \left.\frac{(n)(n+1)(2n+)}{6}\right|_{n=23}$$

Mean of square of first 23 natural numbers

$$= \frac{\text{Sum}}{23} = \frac{(23)(23+1)(46+1)}{6\times 23}$$

$$= 188$$

T8 : Solution

(c)

$$\text{Probability of appearing head} = \frac{1}{2}$$

$$\text{Probability of appearing tail} = \frac{1}{2}$$

Out of 8, probability of getting atleast 5 heads

$$= {}^8C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 + {}^8C_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2$$

$$+ {}^8C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^1 + {}^8C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^0$$

$$= ({}^8C_5 + {}^8C_6 + {}^8C_7 + {}^8C_8)\left(\frac{1}{2}\right)^8 = \frac{93}{256}$$

T9 : Solution

$n = 5$ $$P = P(H) = \frac{1}{2}$$

$$q = 1 - \frac{1}{2} = \frac{1}{2}$$

Probability of setting exactly two heads.

$$P(x = 2) = {}^nC_2\, P^2\, q^{n-1}$$

$$= {}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3$$

$$= \frac{10}{32} = \frac{5}{16}$$

T10 : Solution

(a)

Given $$f(z) = \frac{2z}{(z-1)(z-2)}$$

Given that, $$P(x=1) = \frac{2}{3}P(x=2)$$

$\because$ For Poisson distribution $P(X=x) = \left[\frac{e^{-\lambda}\lambda^x}{x!}\right]$; where λ is mean value of the distribution

$$\therefore \quad \left[\frac{e^{-\lambda}\lambda^1}{1!}\right] = \frac{2}{3}\left[\frac{e^{-\lambda}\lambda^2}{2!}\right]$$

$$\Rightarrow \quad \frac{\lambda}{1} = \frac{2}{3}\frac{\lambda^2}{2}$$

$$\Rightarrow \quad \lambda = 3$$

For Poisson distribution Mean = Variance

Hence, Variance = 3

T11 : Solution

$n = 100$, $P = 0.01$, $\lambda = \text{Mean} = nP = 1$

$$\text{Required probability} = 1 - [P(x=0) + P(x=1) + P(x=2)]$$

$$= 1 - \left[\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!}\right]$$

$$= 1 - \left[e^{-1} + e^{-1} + \frac{e^{-1}}{2}\right]$$

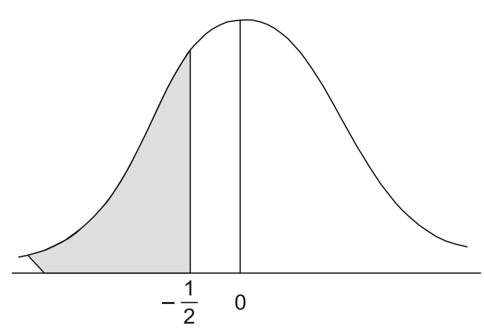
$$= 1 - \frac{5e^{-1}}{2}$$

T12 : Solution

(b)

Here, $\sigma^2 = 4 \quad \Rightarrow \sigma = 2$

$$P(x<0) = p\left(z < \frac{0-\mu}{\sigma}\right) = p\left(z < \frac{0-1}{2}\right) = p\left(z < -\frac{1}{2}\right)$$



Which is the shaded area in the picture and its value is clearly between 0. and 0.5.

T13 : Solution

(a)

$$\text{Given: } f(x) = \frac{1}{5}e^{-x/5}, \; x \geq 0$$

Comparing the general function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}; & x \geq 0 \\ 0; & x < 0 \end{cases} \Rightarrow \lambda = \frac{1}{5}$$

The cumulative distributive function

$$f(a) = 1 - e^{-\lambda a}$$

$$\Rightarrow \quad P(E > 5) = 1 - f(5)$$

$$= 1 - (1 - e^{-\lambda .5})$$

$$= e^{-\frac{1}{5}\times 5} = e^{-1} = 1/e$$

Alternate solution:

$$P(x > 5) = \int_5^{\infty} f(x)\,dx = \int_5^{\infty} \frac{1}{5}e^{-x/5}dx$$

$$= \frac{1}{5}\int_5^{\infty} e^{-x/5}dx$$

$$= \frac{1}{5}\left[-5e^{-x/5}\Big|_5^{\infty}\right] = -1\left[e^{-\infty} - e^{-5/5}\right]$$

$$= e^{-1} = \frac{1}{e}$$

T14 : Solution

(a)

$$\text{Mean}(x_1) = 0.5$$

$$\frac{1}{\lambda_1} = 0.5$$

$$\lambda_1 = \frac{1}{0.5} = 2$$

$$\text{Mean}(x_2) = 0.25$$

$$\frac{1}{\lambda_2} = 0.25$$

$$\lambda_2 = \frac{1}{0.25} = 4$$

$$y = \text{mean}(x_1, x_2)$$

$$\text{Mean}(y) = \frac{1}{\lambda_1 + \lambda_2} = \frac{1}{2+4} = \frac{1}{6}$$

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7 Numerical Methods

Detailed Explanation of Try Yourself Questions

T1 : Solution

Here,
$$x_0 = 2$$
$$f(x) = x^3 - x^2 + 4x - 4$$
$$f'(x) = 3x^2 - 2x + 4$$
$$f(x_0) = f(2) = 8$$
$$f'(x_0) = f'(2) = 12$$

$\Rightarrow$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{8}{12} = \frac{4}{3}$$

T2 : Solution

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$x = 2$,
$$f(x_0) = 2+\sqrt{2}-3 = \sqrt{2}-1$$
$$f'(x) = 1+\frac{1}{2\sqrt{x}}$$
$$f'(x_0) = 1+\frac{1}{2\sqrt{2}}$$

Then,
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2-\frac{\sqrt{2}-1}{1+\frac{1}{2\sqrt{2}}}$$

$\Rightarrow$
$$x_1 = 1.694$$

T3 : Solution

The equation is $f(x) = x^2 - 13 = 0$

Newton-Raphson iteration equation is

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right]$$

$$f(x_0) = x_0^2 - 13$$

$$f'(x_0) = 2x_0$$

$\therefore$ $$x_1 = x_0 - \left[\frac{x_0^2 - 13}{2x_0}\right] = \frac{x_0^2 + 13}{2x_0}$$

put $x_0 = 3.5$ (as given)

$$x_1 = \frac{3.5^2 + 13}{2 \times 3.5} = 3.607$$

$\therefore$ The approximation after one iteration = 3.607

T5 : Solution

(b)

$$u(x_1, x_2) = 10x_2 \sin x_1 - 0.8 = 0$$

$$v(x_1, x_2) = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0$$

The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 10x_2 \cos x_1 & 10 \sin x_1 \\ 10x_2 \sin x_1 & 20x_2 - 10\cos x_1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

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8 Laplace Transforms

Detailed Explanation of Try Yourself Questions

T1 : Solution

(b)

$$L(\sin^2 t)$$

$$L\left(\frac{1-\cos 2t}{2}\right)$$

$$L\left(\frac{1}{2}\right)-L\left(\frac{\cos 2t}{2}\right)$$

$$\frac{1}{2}L(1)-\frac{1}{2}L(\cos 2t)$$

$$\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{2}\cdot\frac{s}{s^2+4} = \frac{s^2+4-s^2}{2s(s^4+4)}$$

$$= \frac{4}{2s(s^2+4)}=\frac{2}{s(s^2+4)}$$

T2 : Solution

(d)

$$L[U(t-a)] = \int_0^\infty e^{-St}\,U(t-a)dt$$

$$= \int_0^a e^{-st}\cdot 0\cdot dt+\int_0^\infty e^{-st}\cdot 1\cdot dt$$

$$= 0+\int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = \frac{e^{-as}}{s}$$

T3 : Solution

(c)

By property of Laplace

$$L(e^{at}t^n) = \frac{\lceil n+1}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}$$

$\because$ $a = -2,\ n = 3$

So the laplace transform will be

$$= \frac{3!}{(s+2)^4} = \frac{6}{(s+2)^4}$$

T4 : Solution

(d)

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Given that,
$$F(s) = \left[\frac{3s+1}{s^3+4s^2+(K-3)s}\right]$$

$$\lim_{t\to\infty} f(t) = 1$$

$\Rightarrow$ $$\lim_{s\to 0} s\left[\frac{3s+1}{s^3+4s^2+(K-3)s}\right] = 1$$

$\Rightarrow$ $$\lim_{s\to 0}\left[\frac{3s+1}{s^2+4s+(K-3)}\right] = 1$$

$\Rightarrow$ $$\frac{1}{K-3} = 1$$

$\Rightarrow$ $$K-3 = 1$$

$\Rightarrow$ $$K = 4$$

T5 : Solution

(c)

By property of Laplace

$\because$ $$L(\cos at) = \frac{s}{s^2+a^2}$$

$\therefore$ $$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

In the given question $a = 2$

Hence, $$L^{-1}\left(\frac{s^2}{s^2+4}\right) = \cos 2t$$

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10 Partial Differential Equations

Detailed Explanation *of* Try Yourself Questions

T1 : Solution

Given equation is
$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \text{...(1)}$$

Let
$$u(x, y) = XY \quad \text{...(2)}$$

where
$$X = f(x) \text{ and } Y = f(y)$$

$$\frac{\partial u}{\partial x} = X'Y; \quad \frac{\partial u}{\partial y} = XY'$$

$$X'Y = 4XY'$$

$$\frac{X'}{X} = 4, \frac{Y'}{Y} = K$$

$$\frac{X'}{X} = K, \frac{Y'}{Y} = \frac{K}{4}$$

$$X = C_1e^{Kx}; \; Y = C_2e^{Ky/4}$$

$\therefore$
$$u(x, y) = XY$$
$$= C_1C_2e^{Kx}eK^{y/4}$$

Now,
$$u(0, y) = C_1C_2e^{Ky/4} = 8e^{-3y}$$

$$C_1C_2 = 8, \frac{K}{4} = -3$$

$$K = -12$$

$$u(x, y) = 8e^{-12x}e^{-3y}$$

$$u(x, y) = 8e^{-12x-3y}$$

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