Electronics Engineering

Electromagnetics

Comprehensive Theory

with **Solved Examples** and **Practice Questions**

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Electromagnetics

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Electromagnetics

CHAPTER 1

Vector Analysis

1.1 Introduction

This introductory chapter provides an elegant mathematical language in which electromagnetic (EM) theory is conveniently expressed and best understood. The quantities of interest appearing in the study of EM theory can almost be classified as either a scalar or a vector.

Quantities that can be described by a magnitude alone are called **scalars**. Distance, temperature, mass etc. are examples of scalar quantities. Other quantities, called **vectors**, require both a magnitude and a direction to fully characterize them. Examples of vector quantities include velocity, force, acceleration etc.

In electromagnetics, we frequently use the concept of a **field**. A field is a function that assigns a particular physical quantity to every point in a region. In general, a field varies with both position and time. There are scalar fields and vector fields. Temperature distribution in a room and electric potential are examples of scalar fields. Electric field and magnetic flux density are examples of vector fields.

NOTE: Vectors are denoted by an arrow over a letter ($\vec{A}$) and scalars are denoted by simple letter (A).

1.1.1 Unit Vector

A unit vector $\hat{a}_A$ along $\vec{A}$ is defined as a vector whose magnitude is unity (*i.e.*, 1) and its direction is along $\vec{A}$, that is

$$\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \quad \text{...(1.1)}$$

Thus we can write $\vec{A}$ as

$$\vec{A} = A\,\hat{a}_A = |\vec{A}|\hat{a}_A \quad \text{...(1.2)}$$

Remember: Any vector can be written as product of its magnitude and its unit vector.

1.1.2 Vector Addition and Subtraction

Two vectors $\vec{A}$ and $\vec{B}$ can be added together to give another vector $\vec{C}$; that is,

$$\vec{C} = \vec{A}+\vec{B} \quad \text{...(1.3)}$$

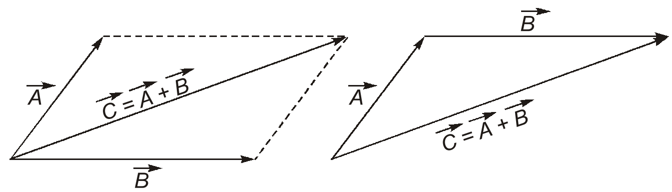


***Figure.* 1.1:** *Vector addition (a) parallelogram rule, (b) head-to-tail rule.*

NOTE

- $\vec{A}+\vec{B}=\vec{B}+\vec{A}$ (Commutative law)
- $(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C})$ (Associative law)

Vector subtraction is similarly carried out as

$$\vec{D} = \vec{A}-\vec{B}=\vec{A}+(-\vec{B}) \quad \text{...(1.4)}$$

Remember: Graphically, vector addition and subtraction are obtained by either the parallelogram rule or the head-to-tail rule as portrayed in figure 1.1.

NOTE

- $k(\vec{A}+\vec{B})=k\vec{A}+k\vec{B}$ (Distributive law)
- $\dfrac{\vec{A}+\vec{B}}{k}=\dfrac{1}{k}\vec{A}+\dfrac{1}{k}\vec{B}$

1.1.3 Position and Distance Vectors:

A point P in Cartesian coordinates may be represented by (x, y, z).

The position vector $\vec{r}_p$ (or radius vector) of point P is defined as the directed distance from origin O to P.

$$\vec{r}_p = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z \quad \text{...(1.5)}$$

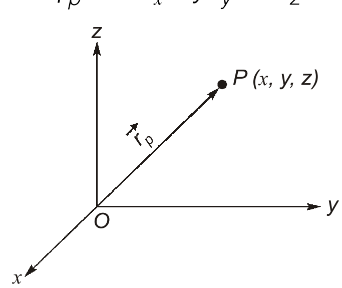


Figure 1.2: *Illustration of position vector* $\vec{r}_p = x\hat{a}_x + \hat{y}a_y + z\hat{a}_z$

The distance vector is the displacement from one point to another.

Consider point P with position vector $\vec{r}_p$ and point Q with position vector $\vec{r}_q$. The displacement from P to Q is written as

$$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p \quad \text{...(1.6)}$$

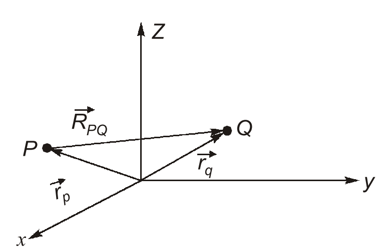


Figure 1.3: *Vector distance* $\vec{R}_{PQ}$

Example 1.1 Point P and Q are located at (0, 2, 4) and (–3, 1, 5). Calculate

(a) The position vector P

(b) The distance vector from P to Q

(c) The distance between P and Q

(d) A vector parallel to PQ with magnitude of 10.

Solution:

(a) $$\vec{r}_P = 0\hat{a}_x + 2\hat{a}_y + 4\hat{a}_z = 2\hat{a}_x + 4\hat{a}_z$$

(b) $$\vec{R}_{PQ} = \vec{r}_q - \vec{r}_p = (-3, 1, 5) - (0, 2, 4) = (-3, -1, 1)$$

$$= -3\hat{a}_x - \hat{a}_y + \hat{a}_z$$

(c) The distance between P and Q is the magnitude of $\vec{R}_{PQ}$; that is

$$d = |\vec{R}_{PQ}| = \sqrt{9+1+1} = 3.317$$

(d) Let the required vector be $\vec{A}$, then

$$\vec{A} = A\hat{a}_A$$

where $A = 10$ is magnitude of $\vec{A}$

and $$\hat{a}_A = \frac{\vec{R}_{PQ}}{|\vec{R}_{PQ}|} = \pm\frac{(-3,-1,1)}{3.317}$$

then $$\vec{A} = \pm\frac{10(-3,-1,1)}{3.317} = \pm(-9.045\,\hat{a}_x - 3.015\,\hat{a}_y + 3.015\,\hat{a}_z)$$

1.1.4 Vector Multiplication

When two vectors are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication.

1. Scalar (or dot) product : $\vec{A} \cdot \vec{B}$
2. Vector (or cross) product : $\vec{A} \times \vec{B}$

Multiplication or three vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ can result in either

3. Scaler triple product : $\vec{A} \cdot (\vec{B} \times \vec{C})$
4. Vector triple product : $\vec{A} \times (\vec{B} \times \vec{C})$

Dot Product:

The dot product, or the scalar product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A} . \vec{B}$ is defined geometrically as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle between them.

Thus $$\vec{A} \cdot \vec{B} = A\,B\cos\theta_{AB} \quad ...(1.7)$$

Where θ_{AB} is the smaller angle between $\vec{A}$ and $\vec{B}$. The result of $\vec{A} . \vec{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign.

If $$\vec{A} = (A_x, A_y, A_z)$$

and $$\vec{B} = (B_x, B_y, B_z)$$

then $$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad ...(1.8)$$

NOTE: Two vectors $\vec{A}$ and $\vec{B}$ are said to be orthogonal (or perpendicular) with each other if $\vec{A}.\vec{B} = 0$

The dot product obeys the following:

Law	Expression	
Cumulative	$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$	...(1.9)
Distributive	$\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$	...(1.10)
	$\vec{A}.\vec{A} = \lvert\vec{A}\rvert^2 = \lvert A\rvert^2$	...(1.11)

Also note that:

$$\hat{a}_x\cdot\hat{a}_y = \hat{a}_y\cdot\hat{a}_z = \hat{a}_z\cdot\hat{a}_x = 0 \quad ...(1.12)$$

$$\hat{a}_x\cdot\hat{a}_x = \hat{a}_y\cdot\hat{a}_y = \hat{a}_z\cdot\hat{a}_z = 1 \quad ...(1.13)$$

Cross Product:

The cross product of two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A}\times\vec{B}$, is a vector quantity whose magnitude is the area of the parallelepipedformed by $\vec{A}$ and $\vec{B}$ and is in the direction of advance of the right-handed screw as $\vec{A}$ is turned into $\vec{B}$.

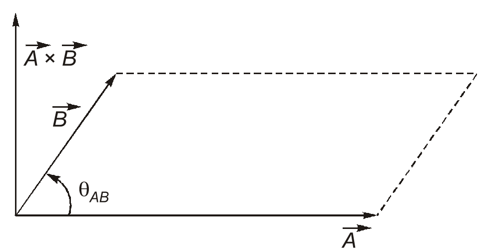


***Figure 1.4:** The cross product of $\vec{A}$ and $\vec{B}$ is a vector with magnitude equal to the area of parallelogram and the direction as indicated*

Thus
$$\vec{A}\times\vec{B} = AB\sin\theta_{AB}\,\hat{a}_n \quad ...(1.14)$$

where $\hat{a}_n$ is a unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$.

The vector multiplication of equation (1.14) is called **cross product** due to the cross sign. It is also called **vector product** because the result is a vector.

If $\vec{A} = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$ then :

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ...(1.15)$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{a}_x + (A_zB_x - A_xB_z)\hat{a}_y + (A_xB_y - A_yB_x)\hat{a}_z \quad ...(1.16)$$

Which is obtained by 'crossing' terms in cyclic permutation, hence the name cross product.

Note that the cross product has the following properties

1. It is not commutative:

$$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A} \quad ...(1.17)$$

NOTE: $\vec{A}\times\vec{B}\neq\vec{B}\times\vec{A}$

2. It is not associative:

$$\vec{A}\times(\vec{B}\times\vec{C})\neq(\vec{A}\times\vec{B})\times\vec{C} \quad ...(1.18)$$

3. It is distributive:

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C} \quad ...(1.19)$$

NOTE: $\vec{A}\times\vec{A}=0$

4. Also note that

$$\hat{a}_x\times\hat{a}_y = \hat{a}_z \quad ...(1.20)$$

$$\hat{a}_y\times\hat{a}_z = \hat{a}_x \quad ...(1.21)$$

$$\hat{a}_z\times\hat{a}_x = \hat{a}_y \quad ...(1.22)$$

NOTE: If $\vec{A}\times\vec{B}=0$, then $\sin\theta_{AB}=0°$ or $180°$; this shows that $\vec{A}$ and $\vec{B}$ are parallel or antiparallel to each other

Scalar Triple Product:

Given three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define scalar triple product as,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\,.\,(\vec{C}\times\vec{A}) = \vec{C}\,.\,(\vec{A}\times\vec{B}) \quad ...(1.23)$$

If $\vec{A}=(A_x, A_y, A_z)$, $\vec{B}=(B_x, B_y, B_z)$ and $\vec{C}=(C_x, C_y, C_z)$, then $\vec{A}\,.\,(\vec{B}\times\vec{C})$ is the volume of a parallelopiped having $\vec{A}$, $\vec{B}$, and $\vec{C}$ as edges and is easily obtained by finding the determinant of the 3×3 matrix formed by $\vec{A}$, $\vec{B}$, and $\vec{C}$; that is

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \quad ...(1.24)$$

Since the result of this vector multiplication is scalar these two equations are called the scalar triple product.

Vector Triple Product:

For vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, we define the vector triple product as

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\,.\,\vec{C})-\vec{C}(\vec{A}.\vec{B}) \quad ...(1.25)$$

This is obtained using the "bac-cab" rule.

Example 1.2 Given vectors $\vec{A}=3\hat{a}_x+4\hat{a}_y+\hat{a}_z$ and $\vec{B}=2\hat{a}_y-5\hat{a}_z$, find the angle between $\vec{A}$ and $\vec{B}$

Solution:

The angle θ_{AB} can be found by using either dot product or cross product

$$\vec{A}\cdot\vec{B} = (3, 4, 1)\,.\,(0, 2, -5) = 0+8-5 = 3$$

$$|\vec{A}| = \sqrt{3^2+4^2+1^2} = \sqrt{26}$$

$$|\vec{B}| = \sqrt{0^2+2^2+5^2} = \sqrt{29}$$

$$\cos\theta_{AB} = \frac{\vec{A}\,.\,\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{3}{\sqrt{(26)\times(29)}} = 0.1092$$

$$\theta_{AB} = \cos^{-1}(0.01092) = 83.73°$$

Alternatively:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix} = (-20-2)\,\hat{a}_x + (0+15)\,\hat{a}_y + (6-0)\,\hat{a}_z$$

$$= (-22, 15, 6)$$

$$|\vec{A}\times\vec{B}| = \sqrt{(-22)^2+(15)^2+(6)^2} = \sqrt{745}$$

$$\sin\theta_{AB} = \frac{\vec{A}\times\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{\sqrt{745}}{\sqrt{(26)\times(29)}} = 0.994$$

$$\theta_{AB} = \sin^{-1}(0.994) = 83.73°.$$

Example 1.3 Three field quantities are given by $\vec{P} = 2\hat{a}_x - \hat{a}_z$ and $\vec{Q} = 2\hat{a}_x - \hat{a}_y + 2\hat{a}_z$, $\vec{R} = 2\hat{a}_x - 3\hat{a}_y + \hat{a}_z$. Determine:

(a) $(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q})$
(b) $\vec{Q}\,.\,(\vec{R}\times\vec{P})$
(c) $\vec{P}.\,(\vec{Q}\times\vec{R})$
(d) $\sin\theta_{QR}$
(e) $\vec{P}\times(\vec{Q}\times\vec{R})$
(f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$
(g) The component of $\vec{P}$ along $\vec{Q}$

Solution:

(a)
$$(\vec{P}+\vec{Q})\times(\vec{P}-\vec{Q}) = 2(\vec{Q}\times\vec{P})$$

$$= 2\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= 2(1-0)\,\hat{a}_x + 2\,(4+2)\,\hat{a}_y + 2(0+2)\,\hat{a}_z$$

$$= 2\hat{a}_x + 12\hat{a}_y + 4\hat{a}_z$$

(b)
$$\vec{Q}.(\vec{R}\times\vec{P}) = (2, -1, 2)\,.\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix}$$

$$= (2, -1, 2).\,(3, 4, 6)$$

$$= 6 - 4,\ 12 = 14$$

Alternatively:
$$\vec{Q}.(\vec{R}\times\vec{P}) = \begin{vmatrix} 2 & -1 & 2 \\ 2 & -3 & 1 \\ 2 & 0 & -1 \end{vmatrix} = 14$$

(c) $\vec{P}.(\vec{Q}\times\vec{R}) = \vec{Q}.(\vec{R}\times\vec{P}) = 14$

(d) $$\sin\theta_{QR} = \frac{|\vec{Q}\times\vec{R}|}{|\vec{Q}|\,|\vec{R}|} = \frac{\sqrt{45}}{3\sqrt{14}} = 0.5976$$

(e) $$\vec{P}\times(\vec{Q}\times\vec{R}) = \vec{Q}(\vec{P}.\vec{R}) - \vec{R}(\vec{P}.\vec{Q})$$
$$= (2, -1, 2)\,(4 + 0 - 1) - (2, -3, -1)\,(4 + 0 - 2)$$
$$= (2, 3, 4)$$
$$= 2\hat{a}_x + 3\hat{a}_y + 4\hat{a}_z$$

(f) A unit vector perpendicular to both $\vec{Q}$ and $\vec{R}$ is given by

$$\hat{a}_n = \pm\frac{\vec{Q}\times\vec{R}}{|\vec{Q}\times\vec{R}|} = \frac{\pm(5,2,-4)}{\sqrt{45}}$$
$$= \pm\,(0.745, 0.298, -0.596)$$
$$\hat{a}_n = \pm\,(0.745\,\hat{a}_x + 0.298\,\hat{a}_y - 0.596\,\hat{a}_z)$$

Note that, $|\hat{a}_n| = 1,\ \hat{a}_n.\vec{Q} = \hat{a}_n.\vec{R} = 0$

The component of $\vec{P}$ along $\vec{Q}$ is

$$\vec{P}_Q = |\vec{P}|\cos\theta_{PQ}\,\hat{a}_Q$$
$$= (\vec{P}.\hat{a}_Q)\hat{a}_Q$$
$$= \frac{(\vec{P}.\vec{Q})\vec{Q}}{|\vec{Q}|^2} = \frac{(4+0-2)(2,-1,2)}{(4+1+4)}$$
$$= \frac{2}{9}(2,-1,2)$$
$$= 0.4444\,\hat{a}_x - 0.2222\,\hat{a}_y + 0.4444\,\hat{a}_z$$

1.2 Coordinate Systems

A coordinate system defines points of reference from which specific vector directions may be defined.

Depending on the geometry of the application, one coordinate system may lead to more efficient vector definitions than others. The three most commonly used co-ordinate systems used in the study of electromagnetics are rectangular coordinates (or Cartesian coordinates), cylindrical coordinates and spherical coordinates.

NOTE: An orthogonal system is one in which the coordinates are mutually perpendicular

1.2.1 Cartesian Coordinates

A vector $\vec{A}$ in Cartesian (other wise known as rectangular) coordinates can be written as

(A_x, A_y, A_z) or $A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$ ...(1.26)

Where a_x, a_y, a_z are unit vectors along the x, y and z directions

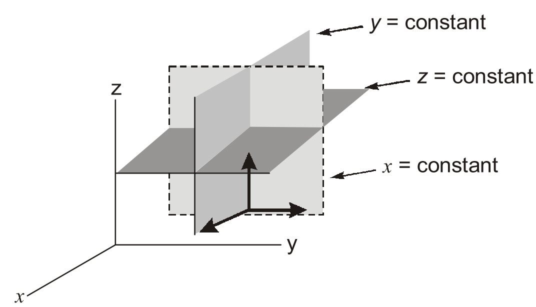


Figure 1.5: *A point in Cartesian coordinates is defined by the intersection of the three planes: x = constant, y = constant, z = constant. The three unit vectors are normal to each of the three surfaces.*

The ranges of the variables are:

$$-\infty \le x \le +\infty \quad \text{...(1.27 a)}$$
$$-\infty \le y \le +\infty \quad \text{...(1.27 b)}$$
$$-\infty \le z \le +\infty \quad \text{...(1.27 c)}$$

1.2.2 Cylindrical Coordinates

The cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z) and is as shown in Fig 1.6. Observe Fig. 1.6 closely and note how we define each space variable; ρ is the radius of the cylinder passing through P or the radial distance from the z-axis; ϕ, called the azimuthal angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are:

$$0 \le \rho \le \infty \quad \text{...(1.28)}$$
$$0 \le \phi \le 2\pi$$
$$-\infty \le z \le +\infty$$

A vector $\vec{A}$ in cylindrical coordinates can be written as

$$(A_\rho, A_\phi, A_z) \text{ or } A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z \quad \text{...(1.29)}$$

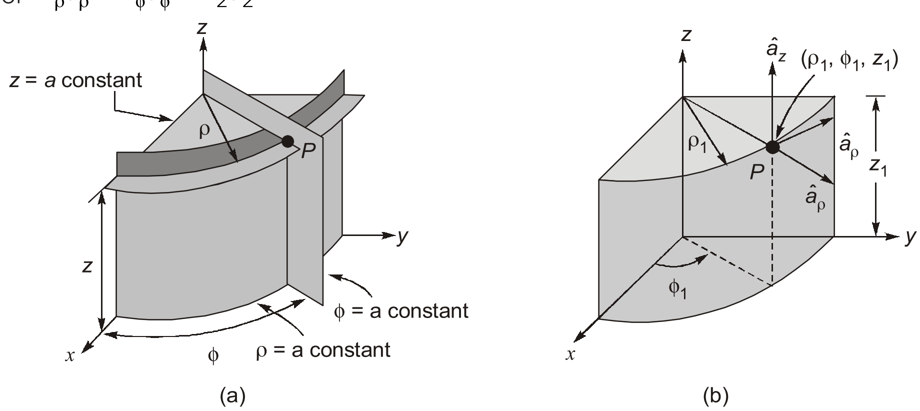


Figure 1.6: *(a) The point is defined by the intersection of the cylinder and the two planes. (b) Point P and unit vectors in the cylindrical coordinate system.*

Notice that the unit vectors $\hat{a}_\rho, \hat{a}_\phi$ and $\hat{a}_z$ are mutually perpendicular because our coordinate system is orthogonal.

$$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0 \quad \text{...(1.30)}$$

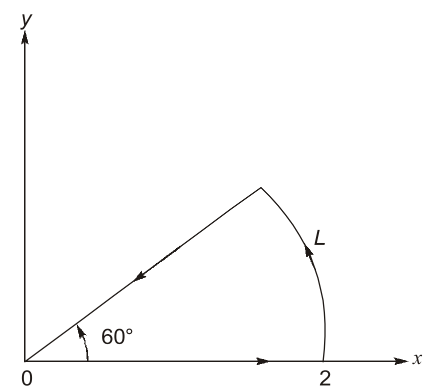
Student's Assignments 2

Q.1 Find the area of the curved surface of a right circular cylinder of radius r and height h using cylindrical coordinates.

Q.2 Calculate the volume in spherical coordinates defined by

$$1 \le r \le 2 \text{ m},\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi \le \frac{\pi}{2}$$

Q.3 Calculate the circulation of $A = \rho\cos\phi\hat{a}_\rho + z\sin\phi\hat{a}_z$ around the edge L of the wedge defined by $0 < \rho < 2$, $0 \le \phi \le 60°$, $z = 0$ and shown in figure.

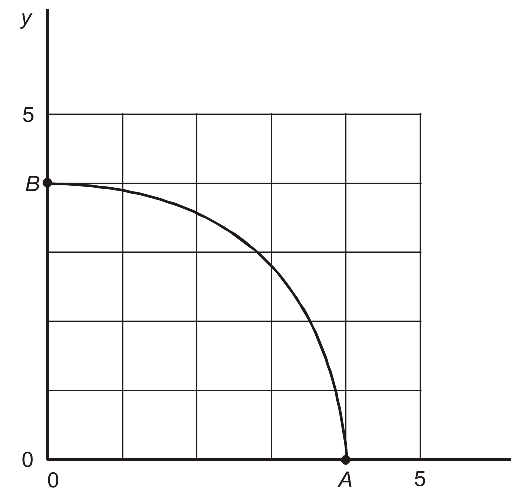


Q.4 Consider a vector $\vec{A} = x^2yz\hat{a}_x + xy^2z\hat{a}_y + xyz^2\hat{a}_z$ evaluate the surface integral for a surface enclosing a unit cube defined by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. Verify the result using divergence theorem.

Q.5 Consider a scalar field given by $V = xy^2z^3$. Find the rate of change of V in space at (1, 1, 1).

Q.6 Calculate the work ΔW required to move the cart along the circular path from point A to point B if the force field is

$$\vec{F} = 3xy\hat{a}_x + 4xy\hat{a}_y$$



ANSWERS

1. $(2\pi rh)$ 2. $\left(\frac{7\pi}{6}\text{m}^3\right)$ 3. (1) 4. (3/4)

5. $\hat{a}_x + 2\hat{a}_y + 3\hat{a}_z$ 6. $-64 + 16\pi$