POSTAL Study Package 2021

Production and Industrial Engineering

Objective Practice Sets

Engineering Mathematics

Contents

MADE EASY
Publications

Calculus

Q.1 In the Mean value theorem $f(b) - f(a) = (b - a)f'(c)$, the value of c lying between a and b, if $f(x) = x(x - 1)(x - 2)$, $a = 0$ and $b = 1/2$ is _____.

Q.2 Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Q.3 A rectangular sheet of a metal of length 6 metres and width 2 metres is given. Four equal squares are removed from the corners. The sides of this sheet are now turned up to form an open rectangular box. The approximately height of the box in cm such that the volume of the box is maximum is _____.

Q.4 Evaluate : $\int_0^{\pi} \frac{\sqrt{(1-\cos x)}}{1+\cos x}\sin^2 x\,dx$

Q.5 Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.

(a) $3\pi a^2$ (b) $2\pi a^2$
(c) $4\pi a^2$ (d) $6\pi a^2$

Q.6 Evaluate $\iint_R x^2\,dxdy$ where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$ will be ____.

Q.7 The value of $\iint_R (x+y)\,dydx$, R is the region bounded by $x = 0$, $x = 2$, $y = x$, $y = x + 2$.

Q.8 The value of $\int_0^{\pi}\int_0^{a} r^3 \sin\theta\cos\theta\,dr\,d\theta$ is _____.

Q.9 The area lying inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = a$ is _____.

(a) $\frac{a}{4}(\pi+8)$ (b) $\frac{a^2}{6}(\pi+8)$
(c) $\frac{a^2}{12}(\pi+8)$ (d) $\frac{a^3}{12}(\pi+8)$

Q.10 The value of $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{(1-x^2)(1-y)^2}}$ is

(a) $\frac{\pi^2}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi^2}{4}$ (d) $\frac{\pi}{2}$

Q.11 $\int_0^{\pi/2} (\cos^3 x)\,dx =$

(a) 3/2 (b) 2/3
(c) 8/9 (d) 8/13

Q.12 If $x = e^{y+e^{y+e^{y+\dots}}}$ then $\frac{dy}{dx}$ is

(a) $(1 - x)$ (b) $(1 - x)/x$
(c) $1/x$ (d) $x/(1 - x)$

Q.13 Let $f(x) = x^2 - x - 2$ then which of the following is true?

(a) $f(x)$ has minima at $\frac{1}{2}$
(b) $f(x)$ has maxima at $\frac{1}{2}$
(c) $f(x)$ has minima at -1
(d) $f(x)$ has maxima at 1

Q.14 The number of solutions for $x = \cos x$ are _______

Q.15 The value of $\int_{-5}^{5} |x+1|\,dx$ is __________.

Q.16 If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then $\lim_{x\to a}\frac{g(x)f(a) - g(a)f(x)}{x-a}$ is ________

Answers Calculus

1. (0.236)	2. (0)	3. (45)	4. (3.77)	5. (a)	6. (448)	7. (12)	8. (0)
9. (a)	10. (c)	11. (b)	12. (b)	13. (a)	14. (1)	15. (26)	16. (5)
17. (a)	18. (1.5)	19. (c)	20. (36)	21. (2)	22. (d)	23. (18)	24. (b)
25. (a)	26. (b)	27. (d)	28. (c)	29. (d)	30. (a)	31. (a)	32. (d)
33. (b)	34. (c)	35. (a)					

Explanations Calculus

1. (0.236)

$$f(a) = 0$$

$$f(b) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{8}$$

$$f'(x) = 3x^2 - 6x + 2$$

$$f'(c) = 3c^2 - 6c + 2$$

Substituting in (i),

$$\frac{3}{8} - 0 = \left(\frac{1}{2} - 0\right)(3c^2 - 6c + 2)$$

or $12c^2 - 25c + 5 = 0$

whence $$c = \frac{24 \pm \sqrt{(24)^2 - 12 \times 5 \times 4}}{24}$$

$$= 1 \pm 0.764$$

$$= 1.764;\ 0.236$$

Hence, $c = 0.236$, since it only lies between 0 and $\frac{1}{2}$.

2. (0)

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x - \sin x}{x \sin x}$$

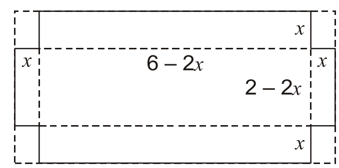
$$= \lim_{x\to 0}\frac{1 - \cos x}{x \sin x + \sin x} \qquad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\sin x}{x(-\sin x) + \cos x + \cos x}$$

$$= \frac{0}{0 + 1 + 1} = 0$$

3. (45)

Let the side of each of the squares cut off be x m so that the height of the box is x m and the sides of the base are $6 - 2x$, $2 - 2x$ m.



$\therefore$ Volume V of the box

$$= x(6 - 2x)(2 - 2x)$$

$$= 4(x^3 - 4x^2 + 3x)$$

Then, $$\frac{dV}{dx} = 4(3x^2 - 8x + 3)$$

For V to be maximum or minimum, we must have

$$\frac{dV}{dx} = 0, \text{ i.e., } 3x^2 - 8x + 3 = 0$$

$\therefore$ $$x = \frac{8 \pm \sqrt{[64 - 4 \times 3 \times 3]}}{6}$$

$$= 2.2 \text{ or } 0.45 \text{ m}$$

The value $x = 2.2$ m is inadmissible, as no box is possible for this value.

Also $\frac{d^2V}{dx^2} = 4(6x - 8)$, which is –ve for $x = 0.45$ m.

Hence, the volume of the box is maximum when its height is 45 cm.

4. (3.77)

Putting $x = 2\theta$, we get

$$= \int_0^{\pi} \frac{\sqrt{(1 - \cos x)}}{1 + \cos x} \sin^2 x\, dx$$

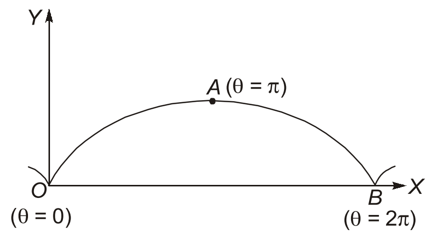
$$= \int_0^{\pi/2} \frac{\sqrt{(1 - \cos 2\theta)}}{1 + \cos 2\theta} \sin^2 2\theta \cdot 2d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sqrt{2}\sin\theta}{2\cos^2\theta} \cdot (2\sin\theta\cos\theta)^2 d\theta$$

$$= 4\sqrt{2}\int_0^{\pi/2} \sin^3\theta\, d\theta = 4\sqrt{2} \cdot \frac{2}{3} = \frac{8\sqrt{2}}{3}$$

5. (a)

To describe its first arch, θ varies from 0 to 2π, i.e., x varies from 0 to $2a\pi$.



$\therefore$ Required area

$$= \int_{x=0}^{2\pi a} y\,dx$$

where $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1-\cos\theta)\cdot a(1-\cos\theta)d\theta$$

$$= 2a^2\int_0^{\pi}(1-\cos\theta)^2 d\theta \;=\; 8a^2\int_0^{\pi}\sin^4\frac{\theta}{2}d\theta$$

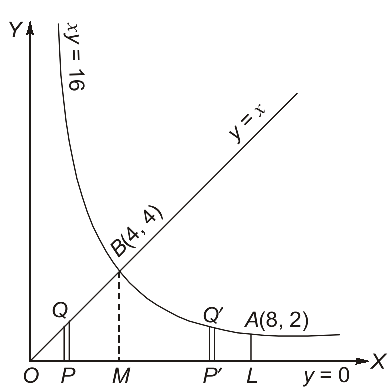
$$= 16a^2\int_0^{\pi/2}\sin^4\phi\, d\phi\,,$$

Putting $\dfrac{\theta}{2} = \phi$ so that $d\theta = 2d\phi$.

$$= 16a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

6. (448)

The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at $A(8, 2)$ while the line $y = x$ intersects this hyperbola at $B(4, 4)$. Figure shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.



$$\therefore \iint_R x^2dxdy = \int_{x \text{ at } 0}^{x \text{ at } M}\int_{y \text{ at } P}^{y \text{ at } Q} x^2dxdy + \int_{x \text{ at } M}^{x \text{ at } L}\int_{y \text{ at } P'}^{y \text{ at } Q'} x^2dxdy$$

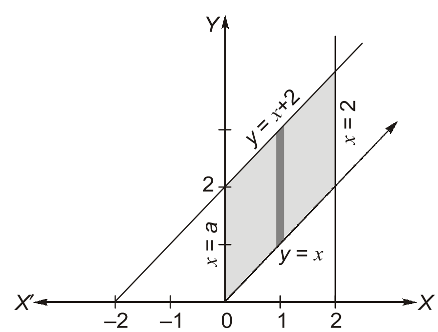
$$= \int_0^4\int_0^x x^2dxdy + \int_4^8\int_0^{16/x} x^2dxdy$$

$$= \int_0^4 x^2dx\,|y|_0^x + \int_4^8 x^2dx\,|y|_0^{16/x}$$

$$= \int_0^4 x^3dx + \int_4^8 16x\,dx$$

$$= \left|\frac{x^4}{4}\right|_0^4 + 16\left|\frac{x^2}{2}\right|_4^8 = 448$$

7. (12)



Let $\quad I = \iint_R (x+y)dydx$

The limits are $x = 0$, $x = 2$, $y = x$ and $y = x + 2$

$$I = \int_0^2 dx\int_x^{x+2}(x+y)dy$$

$$= \int_0^2\left[xy + \frac{y^2}{2}\right]_x^{x+2} dx$$

$$= \int_0^2\left[x(x+2) + \frac{1}{2}(x+2)^2 - x^2 - \frac{x^2}{2}\right]dx$$

$$= \int_0^2\left[x^2 + 2x + \frac{1}{2}(x^2+4x+4) - x^2 - \frac{x^2}{2}\right]dx$$

$$= \int_0^2[2x + 2x + 2]dx$$

$$= 2\int_0^2(2x+1)dx = 2\left[x^2 + x\right]_0^2$$

$$= 2[4 + 2] = 12$$

$$y_2 = \frac{x+8}{2}$$

$$y_1 = \frac{x^2}{8}$$

$$= \int_{-4}^{8}\left(\frac{x+8}{2} - \frac{x^2}{8}\right)dx$$

$$= \left[\frac{x^2}{4} + \frac{8x}{2} - \frac{1}{8}\frac{x^3}{3}\right]_{-4}^{8}$$

= 36 square unit

21. **2 (1.90 to 2.10)**

$$f(x) = x^2 e^{-x}$$

$$f'(x) = 2x \cdot e^{-x} - x^2 e^{-x}$$

$$= x \cdot e^{-x}(2 - x)$$

for maxima or minima

$$f'(x) = 0$$

$$x \cdot e^{-x}(2 - x) = 0$$

$$x = 0, 2$$

$$f''(x) = (x^2 - 4x + 2)e^{-x}$$

$f''(0) = 2$ (> 0) minima

$f''(2) = -2 \cdot e^{-2}$ (< 0) maxima

22. **(d)**

For function to be differentiable i.e. continuous

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$$

$$f(0^-) = \lim_{x \to 0^-} \frac{\sin(3p-1)x}{3x} \times \frac{(3p-1)}{(3p-1)}$$

$$= \lim_{x \to 0^-} \frac{\sin(3p-1)x}{(3p-1)x} \times \frac{(3p-1)}{3} = \frac{(3p-1)}{3}$$

$$f(0^+) = \lim_{x \to 0^+} \frac{\tan(3p+1)x}{2x} \times \frac{(3p+1)}{(3p+1)}$$

$$= \lim_{x \to 0^+} \frac{\tan(3p+1)x}{(3p+1)x} \times \frac{3p+1}{2} = \frac{3p+1}{2}$$

For function to be continuous,

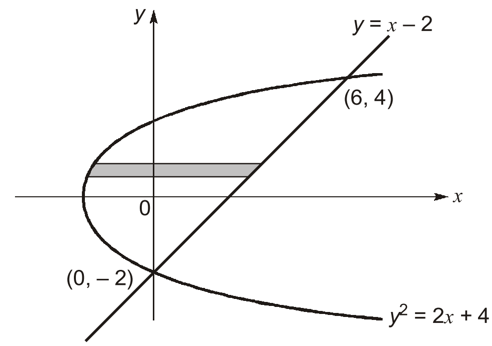
$$\frac{3p-1}{3} = \frac{3p+1}{2}$$

By solving, we get,

$$p = -\frac{5}{3}$$

23. **(18)**

The point of intersection of line and parabolic are (0, –2) and (6, 4).

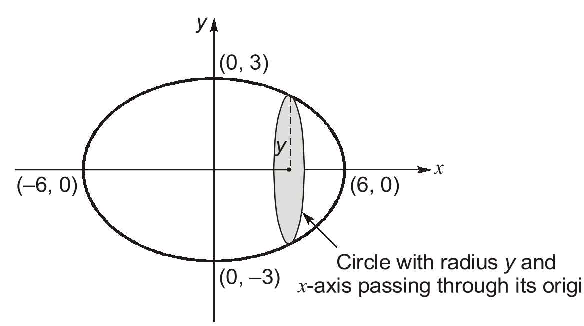


$$\text{Area} = \int_{-2}^{4}\int_{\left(\frac{y^2-4}{2}\right)}^{y+2} dx\,dy = \int_{-2}^{4} x\Big|_{\frac{y^2-4}{2}}^{y+2} dy$$

$$= \int_{-2}^{4}\left(y + 2 - \frac{y^2}{2} + 2\right)dy$$

$$= \left(\frac{y^2}{2} + 4y - \frac{y^3}{6}\right)\Bigg|_{-2}^{4} = 18$$

24. **(b)**



Volume generated

$$= \int_{-6}^{6} \pi y^2 dx = \int_{-6}^{6} \pi\left(\frac{36 - x^2}{4}\right)dx$$

$$= \frac{\pi \times 2}{4}\int_{0}^{6}(36 - x^2)dx = \frac{\pi}{2}\left[36x - \frac{x^3}{3}\right]_0^6$$

$= 72\pi$ unit^3

25. **(a)**

The given curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{6}$$

$$\sqrt{y} = \sqrt{6} - \sqrt{x}$$

$$y = \left(\sqrt{6} - \sqrt{x}\right)^2$$

for $x = 0$, $y = 6$

for $y = 0$, $x = 6$

Now,the area bounded is

$$= \int_0^6 y\,dx = \int_0^6 \left(\sqrt{6}-\sqrt{x}\right)^2 dx$$

$$= \int_0^6 \left(6+x-2\sqrt{6}\sqrt{x}\right) dx$$

$$= \left[6x+\frac{x^2}{2}-2\sqrt{6}\frac{x^{3/2}}{3/2}\right]_0^6$$

$$= \left[36+18-\frac{2\sqrt{6}\times 6\times\sqrt{6}\times 2}{3}\right]$$

$$= 54 - 48 = 6 \text{ unit}^2$$

26. (b)

$$\lim_{x\to 0}\frac{e^{ax}-e^{-ax}}{\ln(1+bx)} = \lim_{x\to 0}\frac{(e^{ax}-e^{-ax})\times 2ax\times b}{2ax\times b\times \ln(1+bx)}$$

$$= \lim_{x\to 0}\left(\frac{e^{ax}-e^{-ax}}{2ax}\right)\times \lim_{x\to 0}\frac{bx}{\ln(1+bx)}\left(\frac{2a}{b}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sinh ax}{ax}\right)\lim_{x\to 0}\frac{bx}{\ln(1+bx)}\left(\frac{2a}{b}\right)$$

$$= 1\times 1\times\frac{2a}{b} = \frac{2a}{b}$$

27. (d)

Given function is

$$y = \frac{1}{x} \qquad \text{[hyperbolic function]}$$

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

Hence, option (d) is correct.

28. (c)

Since $\lim_{x\to 0}(1+0)^{\frac{1}{0}} = 1^{\infty} \rightarrow$ indeterminent

$$f(0) = \lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$$

$$\ln f(0) = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

(Applying L' Hospital rule)

$$\ln f(0) = \lim_{x\to 0}\frac{\cos x}{1+\sin x}\times\frac{1}{1}$$

$$= \frac{1}{1+0} = 1$$

$$\ln f(0) = 1$$

$$f(0) = e^1 = e$$

29. (d)

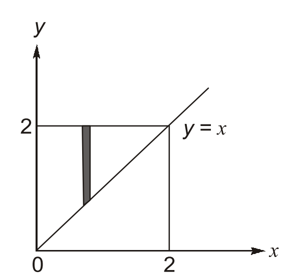
$$I = \int_0^{\frac{\pi}{2}}\frac{\cos x}{\sin x+\cos x}dx$$

$$I = \int_0^{\frac{\pi}{2}}\frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}dx$$

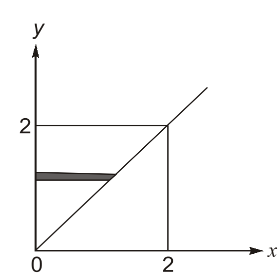
$$2I = \int_0^{\frac{\pi}{2}}\frac{\sin x+\cos x}{\sin x+\cos x}dx$$

$$I = \frac{\pi}{4}$$

30. (a)



$x < y < 2$
$0 < x < 2$



$0 < x < y$
$0 < y < 2$

$$I = \int_0^2\int_0^y f(x,y)\,dx\,dy$$

$$r = p = 0$$

$$q = y$$

$$s = 2$$