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Odisha Public Service Commission

Assistant Executive Engineer

Civil Engineering

Engineering Mechanics

Well Illustrated **Theory** *with*
Solved Examples and **Practice Questions**

MADE EASY
Publications

Engineering Mechanics

Contents

CHAPTER 7 Virtual Work

7.1 Introduction

It has been observed that whenever a force acts on a body, and the body undergoes some displacement, some work is said to be done. Mathematically, if a force (P) acting on a body displaces it through a distance (s), then

$$\text{Work done} = \text{Force} \times \text{Distance} = P \times s$$

But sometimes the body does not move in the direction of force (or in other words, the force does not act in the direction of motion of the body). In such a case,

$$\text{Work done} = \text{Component of the force in the direction of motion} \times \text{Distance}$$
$$= P \cos \theta \times s$$

where θ is the inclination between the line of action of the force and the direction of the motion of the body.

A little consideration will show that

1. If the value of θ is between 0° and 90°, some work is done.
2. If the value of θ is 90°, then no work is done (because cos 90° = 0).
3. If the value of θ is between 90° and 180°, the body will move in the opposite direction and work is called as negative.

7.2 Concept of Virtual Work

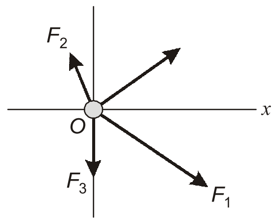
The net work done by applied forces on a body is zero if the body is in static equilibrium, if we assume that the body in equilibrium undergoes an infinitely small imaginary displacement (virtual displacement) some work will be imagined to be done by the applied forces and inertial forces of the body. Such an imaginary work is called virtual work.

If some unknown forces which are acting on the body and keep the body in equilibrium condition, are to be calculated, then the method of virtual work can be used.

Concept of virtual work is useful in finding some unknown forces which are acting on the body and keep the body in static equilibrium condition.

7.3 Principle of Virtual Work

If a system of forces acting on a body or a system of bodies be in equilibrium and the system be imagined to undergo a small displacement consistent with the geometrical conditions, then the algebraic sum of the virtual work done by the forces of the system is zero.



Consider a plane concurrent system of forces F_1, F_2, F_3 ...F_n acting on a particle at O as shown in figure. Let the resultant of forces be R. If system is in equilibrium then

$$R = F_1 + F_2 + F_3 .....F_N = 0$$

Now we impart an infinitely small virtual displacement δr to the particle at O in any arbitrary direction. The work done by the forces F_i through the virtual displacement is

$$\delta U = \delta r \cdot F_1 + \delta r \cdot F_2 + \delta r \cdot F_3 ...... \delta r \cdot F_n$$

$$\delta U = \delta r \cdot (F_1 + F_2 + F_3 ...... F_n)$$

$$= \delta r \cdot R$$

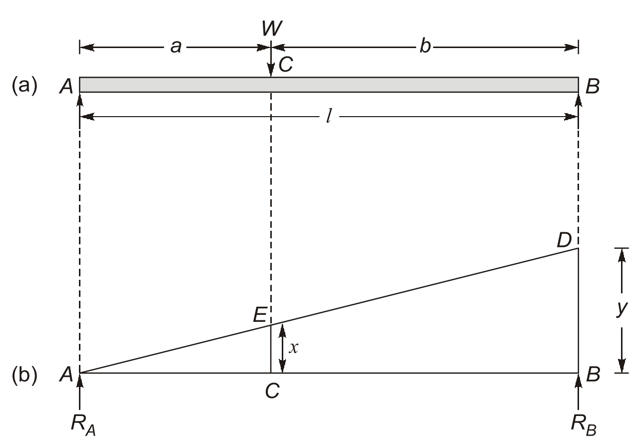
Now if $R = 0$ then $\delta U = 0$. Thus virtual work done by the forces acting on the particle through any virtual displacement is zero.

7.4 Sign Conventions

Though there are different sign conventions for finding out the virtual works done in different books, yet we shall use the following sign conventions, which are internationally recognised.

1. Upward forces are considered as positive, whereas the downwards as negative.
2. Forces acting towards right are considered as positive, whereas those towards left as negative.
3. Forces acting in the clockwise direction are considered as positive, whereas the anticlockwise as negative.
4. Tensile forces are considered as positive whereas the compressive as negative.

7.5 Application of the Principle of Virtual Work on Beams carrying Point Load



Beam carrying point load

Consider a beam AB, simply supported at its supports, and subjected to a point load W at C as shown in figure (a).

Let R_A = Reaction at A, and

R_B = Reaction at B

First of all, let us assume the beam to be hinged at A. Now consider an upward virtual displacement (y) of the beam at B. This is due to the reaction at B acting upwards as shown in figure (b). Let x be the upward virtual displacement of the beam at C due to the point load.

Now in two similar triangles ABD and ACE,

$$\frac{x}{y} = \frac{a}{l} \text{ or } x = \frac{ay}{l}$$

∴ Total virtual work done by the two reactions R_A and R_B

$$= +[(R_A \times 0) + (R_B \times y)] = +R_B \times y \quad \text{...(i)}$$

...(Plus sign due to the reactions acting upwards)

and virtual work done by the point load

$$= -W \times x \quad \text{...(ii)}$$

...(Minus sign due to the load acting downwards)

We know that from the principle of virtual work, that algebraic sum of the virtual works done is zero. Therefore,

$$R_B \times y - W \times x = 0$$

or

$$R_B = \frac{W \times x}{y} = \frac{W}{y} \times \frac{a \times y}{l} = \frac{W \times a}{l}$$

Similarly, it can be proved that the vertical reaction at A,

$$R_A = \frac{W \times b}{l}$$

NOTE

1. For the sake of simplicity, we have taken only one point load W at C. But this principle may be extended for any number of loads.
2. The value of reaction at A (i.e., R_A) may also be obtained by subtracting the value of R_B from the downward load W. Mathematically,

$$R_A = W - \frac{Wa}{l} = W\left(1 - \frac{a}{l}\right) = W\left(\frac{l-a}{l}\right) = \frac{Wb}{l}$$

Example - 7.1 A beam AB of span 5 metres is carrying a point laod of 2 kN at a distance 2 metres from A. Determine the beam reactions, by using the principle of the virtual work.

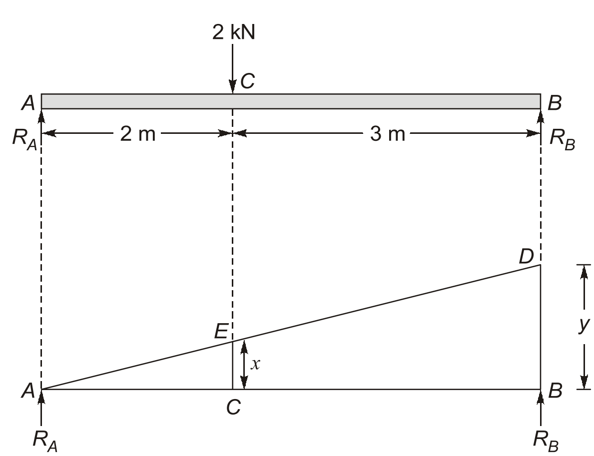
Solution:

Given : Span (l) = 5 m

Point load (W) = 2 kN

and distance between the point load and support

A = 2 m



Let R_A = Reaction at A,

R_B = Reaction at B, and

y = Virtual upward displacement of the beam at B.

From the geometry of the figure, we find that when the virtual upward displacement of the beam at B is y, the virtual upward displacement of the beam at C is $\frac{2y}{5} = 0.4y$ as shown in figure.

∴ Total virtual work done by the two reactions R_A and R_B

$$= [(R_A \times 0) + (R_B \times y)] = R_B \times y \quad \text{...(i)}$$

...(Plus sign due to the reactions acting upwards)

and virtual work done by the point load

$$= -Px = -2 \times 0.4 = -0.8y \quad \text{...(ii)}$$

...(Minus sign due to the load acting downwards)

We know that from the principle of virtual work that algebraic sum of the total virtual works done is zero. Therefore,

$$R_B \times y - 0.8y = 0$$

or $R_B = 0.8\ y/y = 0.8$ kN

and $R_A = 2 - 0.8 = 1.2$ kN

Example - 7.2 A simple supported beam at the ends, 5 m span carries a load of 15 kN at a distance of 2 m from one end. Determine the end reaction using the principle of virtual work.

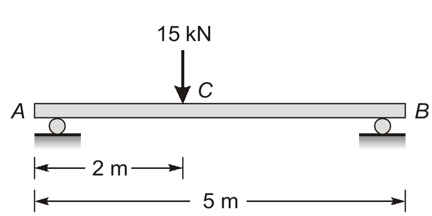
Solution: (?)

As per problem statement the configuration is shown in fig.(a). Assume the virtual displacement given in vertical upper direction at point B is y, then from geometry of fig.(b) displacement y_c at C is

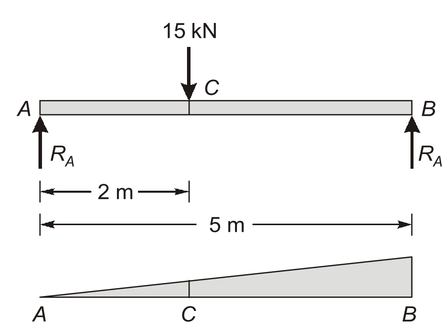
$$y_C = \frac{AC}{AB} y = \frac{2}{5} y.$$

and displacement y_A at A is

$$y_A = 0$$



(a)



(b)

Here beam is in equilibrium under following forces as shown in fig.(b)

(1) 15 kN acting at C, (2) Reaction force R_A acting at A, (3) Reaction force R_B acting at B.

Total work done by these force due to virtual work must be zero.

Thus

$$0 \times R_A - y_C \times 15 + y \times R_B = 0$$

$$0 \times R_A - \frac{2}{5} y \times 15 + y \times R_B = 0$$

or $$-6 + R_B = 0$$

or $$R_B = 6 \text{ kN}$$

The work done by R_A is zero because there is no displacement.

Now resolving forces vertically we get

$$R_A + R_B = 15 \text{ kN}$$

$$R_A = 15 - R_B = 15 - 6 = 9 \text{ kN}$$

7.6 Application of the Principle of Virtual Work on Ladders

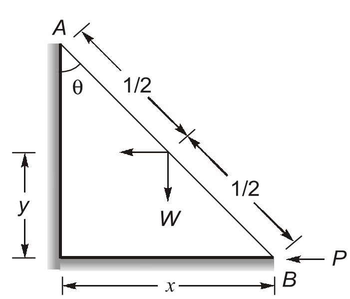
In case of a ladder, its foot moves on the floor towards or away from the wall. It is thus obvious, that no work is done by the normal reaction (R_f) at the foot of the ladder. However, some work is done by the frictional force (F_f) at the foot of the ladder. Similarly, top of the ladder moves up or down along the wall. Thus, no work is done by the nroaml reaction (R_w) at the top of the ladder. However, some work is done by the frictional force at the top of the ladder. This happens when the wall is not smooth, or in other words, the wall has some coefficient of friction.

Now, the virtual works done by the frictional forces at the foot and top of the ladder are found out, and the principle of virtual work is applied as usual.

NOTE:

If the vertical wall is smooth, then there is no frictional force at the top of the ladder. Thus, no work is done at the top of the ladder.

Example - 7.3 A ladder AB of weight W and length l is held in equilibrium by a horizontal force P as shown in the figure. Assuming the ladder to be idealized as a homogeneous rigid bar and the surfaces to be smooth, which one of the following is correct?



(a) $P = \frac{1}{2}W, \tan\theta$

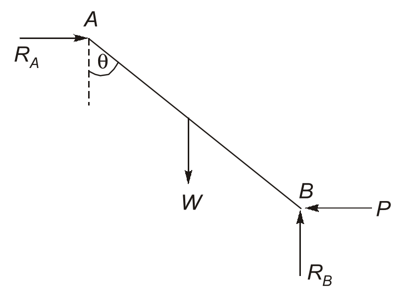
(b) $P = \frac{1}{2}W, \operatorname{cosec}\theta$

(c) $P = \frac{1}{2}W, \cos\theta$

(d) $P = 2W\tan\theta$

Solution:

Free body diagram of ladder is



Using equilibrium equations.

$$R_A = P$$

and

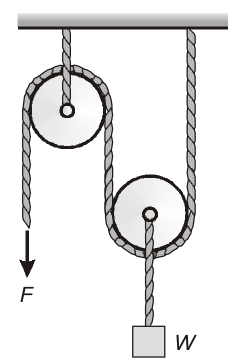
$$R_B = W$$

Taking moment about *B*.

$$R_A \cdot l\cos\theta = W \cdot \frac{l}{2}\sin\theta$$

$$RA = \frac{1}{2}W\tan\theta = P$$

Example - 7.4 A weight of 10 kN is raised by two pulley system as shown in fig. Determine the force *F* required to hold the weight in equilibrium.



Solution:

Assume that *F* goes down through a distance *y*. From the geometry of fig. it may be easily seen that weight moves upward by 1/2 *y* distance.

Using the principal of virtual work

$$F \times y - W \times \frac{y}{2} = 0$$

or

$$F = \frac{W}{2}$$

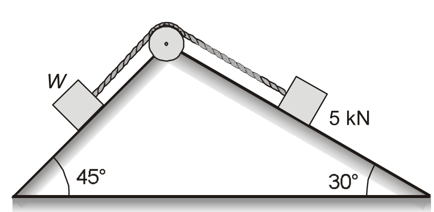
Here

$$W = 10 \text{ kN}$$

Thus

$$F = \frac{10}{2} = 5 \text{ kN}$$

Example - 7.5 A weight of 5 kN resting on a smooth surface inclined 30° to the horizontal is supported by a load *W*, resting on the another smooth surface inclined to horizontal by 45° as shown in fig. Both weight are connected with a string carried over a smooth pulley. Using the principle of virtual work determine the value of *W*.



Solution:

Consider the fig. (a). Assume that W_1 is pulled by *y* along the inclined plane *AC*, then W_2 moves by *y* along *BC*.

Displacement of W_1 in vertical direction = $y \sin \alpha$ upward

Displacement of W_2 in vertical direction = $y \sin \beta$ downward

Thus by principle of virtual work

$$-W_1 \times y \sin\alpha + W_2 \times y \sin\alpha = 0$$

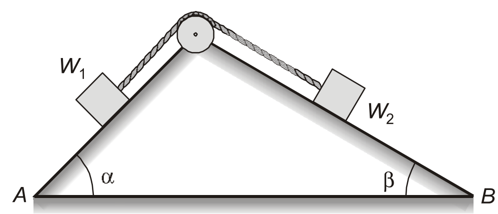
or $$\frac{W_1}{W_2} = \frac{\sin\beta}{\sin\alpha}$$

Here $W_2 = 5$ kN and $W_1 = W$

$\alpha = 45°$ and $\beta = 30°$

Thus $$\frac{W}{5} = \frac{\sin 30°}{\sin 45°}$$

or $$W = 5 \times \frac{1}{2} \times \frac{\sqrt{2}}{1} = \frac{5}{\sqrt{2}} = 3.53 \text{ kN}$$



(a)

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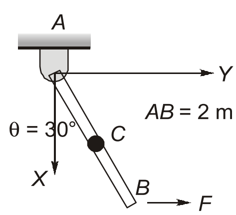
Student's Assignment

Q.1 The term virtual work refers to
(a) Actual work done by virtual forces
(b) Virtual work done by actual forces
(c) Virtual work done by virtual forces
(d) None of these

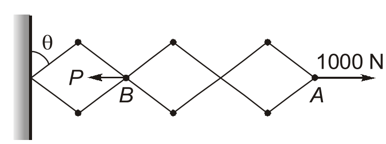
Q.2 A thin rod AB of length 2 m and mass 10 kg is hinged at A and is held in equilibrium, as shown in the given figure, by a horizontal force F acting at B. Which one of the following equations of virtual work will yield value of force F?



(a) $F\delta y_B + 10x_c = 0$ (b) $F\delta y_B = 10\delta x_c$
(c) $F\delta x_B + 10\delta y_c = 0$ (d) $F\delta y_B = 10\delta y_c$

Q.3 The principle of virtual work states that the virtual work is zero for
(a) a body moving with constant linear velocity
(b) a body rotating with constant angular velocity
(c) a body in equilibrium
(d) a body moving with constant linear acceleration

Q.4 A number of rods are hinged together to form three identical rhombuses as shown in figure. If a horizontal force of 1000 N is applied at A, what is the force P required at B for equilibrium?



(a) 3000 N (b) 1000 N
(c) 4000 N (d) 2000 N

ANSWER KEY / STUDENT'S ASSIGNMENT

1. (b) **2.** (c) **3.** (c) **4.** (a)

HINTS & SOLUTIONS / STUDENT'S ASSIGNMENT

2. (c)

The mass 10 kg will act vertically downward from the centre of mass of rod at C. The point C will move upward while the point B moves horizontally due to the force F. The total virtual work done will be zero.

Total virtual work $= F \times \delta x_B + 10\delta y_c = 0$

3. (c)

Principal of virtual work states that total (internal and external) virtual work will be zero if the body is under equilibrium.

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